EX 12.5.6: Using a triple integral, find the volume of the solid E bounded above by the surface $x^2+y^2+z^3=9$ and below by the plane $z=0$.

1^{st}: Intersect surface ($x^2+y^2+z^3=9$) with plane ($z=0$) by plugging the plane (which is simpler) into the surface and rewriting the resulting equation as the canonical form of the appropriate conic section or line:

$$z=0 \implies x^2+y^2+(0)^3=9 \implies x^2+y^2=9 \longleftarrow \text{Circle centered at (0,0) with radius 3}$$

2^{nd}: Project solid E onto the xy-plane, resulting in region D:

The circle $x^2+y^2=9$ is a BC of region D.

Pick a (simple) point inside this circle, say $(x,y)=(-1,0)$.

Plug chosen point into the surface ($x^2+y^2+z^3=9$) & find the corresponding z-coordinate:

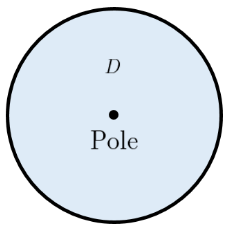
$$(-1)^2+(0)^2+z^3=9 \implies 1+z^3=9 \implies z^3=8 \implies z=2>0$$

Therefore, the surface ($x^2+y^2+z^3=9$) lies **above** the xy-plane ($z=0$) for $x^2+y^2<9$.

Hence, the projection of the solid E onto the xy-plane is $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 9\}$

Observe that region D is the **closed (circular) disk** centered at (0,0) with radius 3.

Since region D is a **disk**, use **polar coordinates** going forward.



3^{rd}: Sketch the region D on the xy-plane, labeling BC's & BP's (in **polar form**) as appropriate:

Notice region D is **radially simple** (r-simple), treating the **pole** ($r=0$) as the **inner BC**.

4^{th}: Setup & compute triple integral for the volume of solid E:

Since solid E is z-simple, expect only one triple integral.

Since region D is r-simple, expect only one double integral afterwards.

$$\begin{aligned}
\text{Volume}(E) &= \iiint_E dV = \iint_D \int_{\text{Btm BS of } E}^{\text{Top BS of } E} dz\ dA = \iint_D \int_0^{\sqrt[3]{9-x^2-y^2}} dz\ dA \\
&= \iint_D \Big[z\Big]_{z=0}^{z=\sqrt[3]{9-x^2-y^2}} dA \overset{FTC}{=} \iint_D \left[\sqrt[3]{9-x^2-y^2}-0\right] dA = \iint_D \sqrt[3]{9-x^2-y^2}\ dA \\
&\overset{POLAR}{=} \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Pole}}^{\text{Outer BC of } D} \sqrt[3]{9-r^2}\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^3 \sqrt[3]{9-r^2}\ r\ dr\ d\theta
\end{aligned}$$

$$CV: \text{Let } u=9-r^2 \implies \left[du=-2r\ dr \iff r\ dr=-\tfrac{1}{2}du\right] \implies u(3)=0 \text{ and } u(0)=9$$

$$\begin{aligned}
&\overset{CV}{=} \int_0^{2\pi}\int_9^0 \sqrt[3]{u}\left(-\frac{1}{2}du\right)d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^9 u^{1/3}\ du\ d\theta = \frac{1}{2}\int_9^{2\pi}\left[\frac{3}{4}u^{4/3}\right]_{u=0}^{u=9} d\theta \overset{FTC}{=} \frac{1}{2}\int_0^{2\pi}\left[\frac{3}{4}(9)^{4/3}-0\right] d\theta \\
&= \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(9^{4/3}\right)\int_0^{2\pi} d\theta = \frac{3}{8}\left(9^{4/3}\right)\Big[\theta\Big]_{\theta=0}^{\theta=2\pi} \overset{FTC}{=} \frac{3}{8}\left(9^{4/3}\right)(2\pi-0) \\
&= \boxed{\frac{3\pi}{4}(9^{4/3})} \approx 44.1097
\end{aligned}$$