EX 12.7.3: Using cylindrical coordinates, compute $I = \iiint_E dV$, where E is the solid bounded above by plane $z = 3$ & below by paraboloid $2z = x^2 + y^2$.

1st, intersect the plane ($z = 3$) with the paraboloid ($2z = x^2 + y^2$) by plugging the plane into the paraboloid and rewriting the resulting equation as the canonical form of the appropriate conic section or line:

$2(3) = x^2 + y^2 \implies x^2 + y^2 = 6 \longleftarrow$ Circle centered at (0,0) with radius $\sqrt{6}$

2nd, write the top & bottom boundary surfaces of solid E **in cylindrical form**:

Use conversion $\{x = r\cos\theta,\ y = r\sin\theta,\ z = z\}$ and write surfaces as $z = f(r, \theta, z)$:

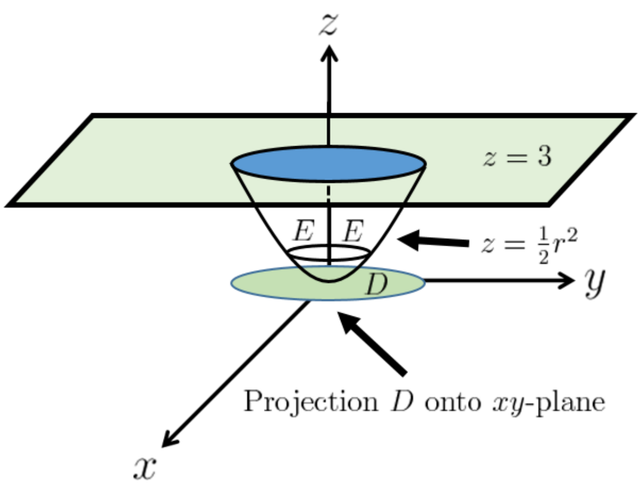
The plane $z = 3$ is already in cylindrical form.

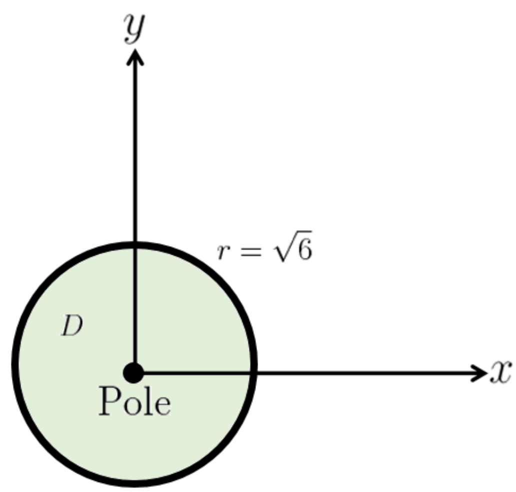
The paraboloid $2z = x^2 + y^2 \implies 2z = (r\cos\theta)^2 + (r\sin\theta)^2 \implies 2z = r^2 \implies z = \frac{1}{2}r^2$ in cylindrical form

3rd, **although not required (and will not be graded on exams)**, it's recommended to sketch solid E in xyz-space. But, it **is required (and will be graded on exams)** to sketch the projection of E onto the xy-plane, AKA region D.

Of course, since cylindrical coordinates will be used, express any BC's of region D in polar form:

$x^2 + y^2 = 6 \iff r = 6$ in polar form





4th, setup & compute the iterated triple integral in cylindrical coordinates:

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\begin{aligned}
I &= \iiint_E dV \\
&\overset{CYL}{=} \int_{\text{Smallest }\theta\text{-value in }D}^{\text{Largest }\theta\text{-value in }D} \int_{\text{Pole}}^{\text{Outer BC in }D} \int_{\text{Btm BS of }E\text{ (in cyl. form)}}^{\text{Top BS of }E\text{ (in cyl. form)}} r\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^{\sqrt{6}}\int_{\frac{1}{2}r^2}^{3} r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{6}} \Big[rz\Big]_{z=\frac{1}{2}r^2}^{z=3} dr\ d\theta \overset{FTC}{=} \int_0^{2\pi}\int_0^{\sqrt{6}} \left[r(3) - r\left(\frac{1}{2}r^2\right)\right] dr\ d\theta = \int_0^{2\pi}\int_0^{\sqrt{6}} \left(3r - \frac{1}{2}r^3\right) dr\ d\theta \\
&= \int_0^{2\pi} \left[\frac{3}{2}r^2 - \frac{1}{8}r^4\right]_{r=0}^{r=\sqrt{6}} d\theta \overset{FTC}{=} \int_0^{2\pi} \left[\left[\frac{3}{2}(\sqrt{6})^2 - \frac{1}{8}(\sqrt{6})^4\right] - \left[\frac{3}{2}(0)^2 - \frac{1}{8}(0)^4\right]\right] d\theta = \int_0^{2\pi} \left[\left(9 - \frac{9}{2}\right) - (0 - 0)\right] d\theta \\
&= \int_0^{2\pi} \frac{9}{2}\ d\theta = \left[\frac{9}{2}\theta\right]_{\theta=0}^{\theta=2\pi} \overset{FTC}{=} \frac{9}{2}(2\pi) - \frac{9}{2}(0) = 9\pi - 0 = \boxed{9\pi}
\end{aligned}
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