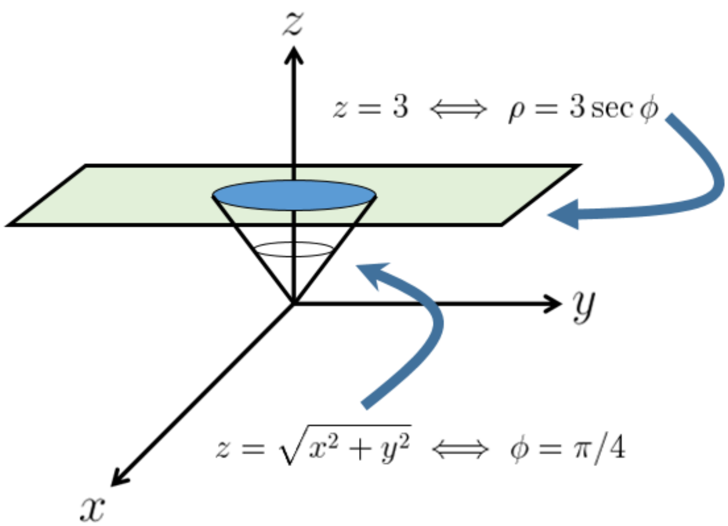
EX 12.7.5: Using spherical coordinates, compute $I = \iiint_E z\, dV$, where E is the solid bounded above by plane $z = 3$ & below by the half-cone $z = \sqrt{x^2 + y^2}$.

1^{st}, **although not required (and will not be graded on exams)**, it's recommended to sketch solid E in xyz-space.

Sketch & label the plane & half-cone in spherical coordinates:



2^{nd}, using "interpretive dance" [as demonstrated in class on 10/28 (Tues)] or otherwise, determine the limits of integration:

$$I = \iiint_E z\, dV$$

$$\overset{SPH}{=} \int_{\text{Smallest } \theta\text{-value in } E}^{\text{Largest } \theta\text{-value in } E} \int_{\text{Smallest } \phi\text{-value in } E}^{\text{Largest } \phi\text{-value in } E} \int_{\text{Inner BS of } E}^{\text{Outer BS of } E} (\rho\cos\phi)\left(\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta\right)$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3\sec\phi} \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{1}{4}\rho^4 \cos\phi\sin\phi\right]_{\rho=0}^{\rho=3\sec\phi} d\phi\, d\theta$$

$$\overset{FTC}{=} \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{1}{4}(3\sec\phi)^4\cos\phi\sin\phi - \frac{1}{4}(0)^4\cos\phi\sin\phi\right] d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{81}{4}\sec^3\phi\sin\phi\, d\phi\, d\theta$$

$$= \frac{81}{4}\int_0^{2\pi}\int_0^{\pi/4}\left(\frac{\sin\phi}{\cos\phi}\right)\sec^2\phi\, d\phi\, d\theta = \frac{81}{4}\int_0^{2\pi}\int_0^{\pi/4} \tan\phi\sec^2\phi\, d\phi\, d\theta$$

CV: Let $u = \tan\phi \implies du = \sec^2\phi\, d\phi \implies u(\pi/4) = \tan(\pi/4) = 1$ and $u(0) = \tan(0) = 0$

$$\overset{CV}{=} \frac{81}{4}\int_0^{2\pi}\int_0^1 u\, du\, d\theta = \frac{81}{4}\int_0^{2\pi}\left[\frac{1}{2}u^2\right]_{u=0}^{u=1} d\theta \overset{FTC}{=} \frac{81}{4}\int_0^{2\pi}\left[\frac{1}{2}(1)^2 - \frac{1}{2}(0)^2\right] d\theta$$

$$= \frac{81}{4}\int_0^{2\pi}\frac{1}{2}\, d\theta = \frac{81}{8}\int_0^{2\pi} d\theta = \frac{81}{8}\Big[\theta\Big]_{\theta=0}^{\theta=2\pi} \overset{FTC}{=} \frac{81}{8}\Big[2\pi - 0\Big] = \boxed{\frac{81}{4}\pi}$$

REMARK 1: The inner BS of solid E is **not** the half-cone, but rather the **origin** which has spherical form $\rho = 0$

REMARK 2: The enforcement of the half-cone as the "lateral BS" of solid E is achieved by restricting ϕ: $0 \le \phi \le \pi/4$

REMARK 3: Remember, projecting solid E onto the xy-plane (resulting in region D) is **useless** with spherical coordinates!