EX 12.8.4: Use the inverse transformation $T^{-1} : \begin{cases} u = xy \\ v = x^2 - y^2 \end{cases}$ to <u>**setup**</u> $\iint_D dA$,

where D is the region **in Quadrant I** bounded by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$ & $x^2 - y^2 = 4$.

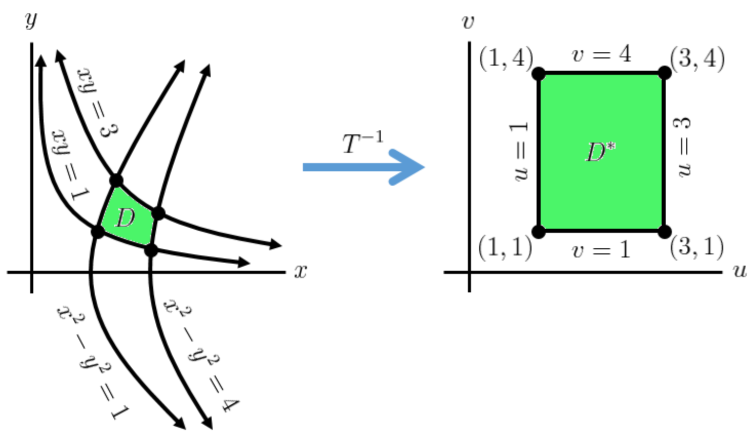
1^{st}, sketch region D in xy-plane & label BC's.

Notice that all four BC's are **hyperbolas**, but only their branches in Quadrant I are necessary to sketch.

Remember for region D, the BP's are **not required** unless they're helpful in labeling the BC's.

Sometimes, the BP's of region D can be really nasty!

Case in point, here the top-left BP of D has coordinates $\left(\frac{\sqrt{1+\sqrt{37}}}{\sqrt{2}}, \frac{3\sqrt{2}}{\sqrt{1+\sqrt{37}}}\right)$ ...yeah...



2^{nd}, apply inverse transformation T^{-1} to each BC of D in xy-plane to obtain the corresponding BC of D^* in uv-plane:

Observe that the BC's of D are of the form: $T_1^{-1}(x, y) = 1$, $T_1^{-1}(x, y) = 3$, $T_2^{-1}(x, y) = 1$, $T_2^{-1}(x, y) = 4$

BC of D		**BC of D^***	**BC of D**		**BC of D^***
$xy = 1$	$\xrightarrow{T^{-1}}$	$u = 1$	$x^2 - y^2 = 1$	$\xrightarrow{T^{-1}}$	$v = 1$
$xy = 3$	$\xrightarrow{T^{-1}}$	$u = 3$	$x^2 - y^2 = 4$	$\xrightarrow{T^{-1}}$	$v = 4$

3^{rd}, sketch region D^* in uv-plane, labeling BC's & BP's. (see above plot)

4^{th}, compute Jacobian:

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix} = -2y^2 - 2x^2 = -2(x^2 + y^2) \implies \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2} \cdot \frac{1}{x^2 + y^2}$$

NOW THE HARD PART: Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ must be in terms of u & v (but how to write $x^2 + y^2$ in terms of u,v ????)

It's not obvious (and you're <u>not</u> expected to remember this slick trick – it won't show up on exams).

The moral here is that sometimes in higher mathematics, you have to "think outside the box":

$$(x^2 + y^2)^2 = x^4 + 2x^2y^2 + y^4 \quad \text{and} \quad (x^2 - y^2)^2 = x^4 - 2x^2y^2 + y^4$$

$$\implies (x^2 + y^2)^2 - (x^2 - y^2)^2 = 4x^2y^2 = 4(xy)^2 \qquad \implies (x^2 + y^2)^2 = 4(xy)^2 + (x^2 - y^2)^2 = 4u^2 + v^2$$

$$\implies x^2 + y^2 = \sqrt{4u^2 + v^2}$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2} \cdot \frac{1}{x^2 + y^2} = -\frac{1}{2} \cdot \frac{1}{\sqrt{4u^2 + v^2}} \implies \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2} \cdot \frac{1}{\sqrt{4u^2 + v^2}}$$

5^{th}, setup double integral in (u, v)-coordinates:

Since region D^* is both V-Simple & H-Simple, either order ($du\,dv$ or $dv\,du$) results in only one double integral:

$$\therefore \iint_D dA \overset{CC}{=} \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \int_{\text{Smallest } v\text{-coord in } D^*}^{\text{Largest } v\text{-coord in } D^*} \int_{\text{Left BC of } D^*}^{\text{Right BC of } D^*} \frac{1}{2\sqrt{4u^2 + v^2}}\, du\, dv = \boxed{\int_1^4 \int_1^3 \frac{1}{2\sqrt{4u^2 + v^2}}\, du\, dv}$$

$$\therefore \iint_D dA \overset{CC}{=} \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA = \int_{\text{Smallest } u\text{-coord in } D^*}^{\text{Largest } u\text{-coord in } D^*} \int_{\text{Bottom BC of } D^*}^{\text{Top BC of } D^*} \frac{1}{2\sqrt{4u^2 + v^2}}\, dv\, du = \boxed{\int_1^3 \int_1^4 \frac{1}{2\sqrt{4u^2 + v^2}}\, dv\, du}$$