EX 13.1.6: Is $f(x,y,z)=\sqrt{1+x^2+y^2+z^2}$ harmonic in $\mathbb{R}^3$? (Justify answer)

Recall that a function is **harmonic** if its **Laplacian equals zero**.

1^{st}, compute the first-order partials of f:

$$f_x = \frac{1}{2}\left(1+x^2+y^2+z^2\right)^{-1/2}\frac{\partial}{\partial x}\left[1+x^2+y^2+z^2\right] = \frac{1}{2}(2x)\left(1+x^2+y^2+z^2\right)^{-1/2} = \frac{x}{\sqrt{1+x^2+y^2+z^2}}$$

$$f_y = \frac{1}{2}\left(1+x^2+y^2+z^2\right)^{-1/2}\frac{\partial}{\partial y}\left[1+x^2+y^2+z^2\right] = \frac{1}{2}(2y)\left(1+x^2+y^2+z^2\right)^{-1/2} = \frac{y}{\sqrt{1+x^2+y^2+z^2}}$$

$$f_z = \frac{1}{2}\left(1+x^2+y^2+z^2\right)^{-1/2}\frac{\partial}{\partial z}\left[1+x^2+y^2+z^2\right] = \frac{1}{2}(2z)\left(1+x^2+y^2+z^2\right)^{-1/2} = \frac{z}{\sqrt{1+x^2+y^2+z^2}}$$

(Notice that the **Ordinary (Calc I) Chain Rule** was used in finding the partial derivatives.)

2^{nd}, compute the non-mixed second-order partials of f:

$$f_{xx} = \frac{\partial}{\partial x}\left[f_x\right] = \frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2+z^2}}\right] = \text{(Use Quotient Rule \& Simplify)} = \frac{1+y^2+z^2}{(1+x^2+y^2+z^2)^{3/2}}$$

$$f_{yy} = \frac{\partial}{\partial y}\left[f_y\right] = \frac{\partial}{\partial y}\left[\frac{y}{\sqrt{1+x^2+y^2+z^2}}\right] = \text{(Use Quotient Rule \& Simplify)} = \frac{1+x^2+z^2}{(1+x^2+y^2+z^2)^{3/2}}$$

$$f_{zz} = \frac{\partial}{\partial z}\left[f_z\right] = \frac{\partial}{\partial z}\left[\frac{z}{\sqrt{1+x^2+y^2+z^2}}\right] = \text{(Use Quotient Rule \& Simplify)} = \frac{1+x^2+y^2}{(1+x^2+y^2+z^2)^{3/2}}$$

3^{rd}, find all points (x,y,z) (if there are any) such that $\nabla^2 f = 0$:

Assume $\nabla^2 f = 0 \quad \forall (x,y,z) \in \mathbb{R}^3$

Then, $f_{xx}+f_{yy}+f_{zz}=0$

$$\implies \frac{3+2x^2+2y^2+2z^2}{(1+x^2+y^2+z^2)^{3/2}} = 0$$

$$\implies 3+2x^2+2y^2+2z^2 = 0$$

$$\implies 2x^2+2y^2+2z^2 = -3$$

$\implies x^2+y^2+z^2 = -3/2 \longleftarrow$ CONTRADICTION! (since a sum of squares cannot equal a negative number!)

Therefore, $\nabla^2 f \neq 0 \;\; \forall (x,y,z) \in \mathbb{R}^3$

Therefore, since $\nabla^2 f$ is never zero, $\boxed{f \text{ is NOT harmonic in } \mathbb{R}^3}$