EX 13.2.3: Compute $\int_C \Big(-xy^2\ dx + x^2\ dy\Big)$, where C is the path from $(-1,1)$ to $(0,1)$ along line $y=1$ and then from $(0,1)$ to $(1,0)$ along quarter-circle $x^2+y^2=1$.

Path C is too hard to parametrize! Therefore, subdivide C:

$C = C_1 \cup (-C_2)$, where
C_1 is the path from $(-1,1)$ to $(0,1)$ along line $y=1$
C_2 is the path from $(1,0)$ to $(0,1)$ along quarter-circle $x^2+y^2=1$ in Quadrant I

Parameterize subpaths C_1 & C_2:

$$C_1 : \begin{cases} x(t) = -1 + [0-(-1)]t \\ y(t) = 1 + (1-1)t \\ t \in [0,1] \end{cases} \overset{SIMPLIFY}{\Longrightarrow} C_1 : \begin{cases} x(t) = t-1 \\ y(t) = 1 \\ t \in [0,1] \end{cases}$$

$$C_2 : \begin{cases} x(t) = \cos t \\ y(t) = \sin t \\ t \in [0,\pi/2] \end{cases}$$ $\longleftarrow$ This parametrization of C_2 is the simplest but is in the CCW orientation! [from (1,0) to (0,1)]

To remedy this, realize that $-C_2$ is in the desired CW orientation. [from (0,1) to (1,0)]

Express dx & dy in terms of t & dt:

$$C_1 : x(t) = t-1 \implies x'(t) = \tfrac{d}{dt}[t-1] = 1 \implies dx = x'(t)\ dt = dt$$
$$C_1 : y(t) = 1 \implies y'(t) = \tfrac{d}{dt}[1] = 0 \implies dy = y'(t)\ dt = 0\ dt$$

$$C_2 : x(t) = \cos t \implies x'(t) = \tfrac{d}{dt}[\cos t] = -\sin t \implies dx = x'(t)\ dt = -\sin t\ dt$$
$$C_2 : y(t) = \sin t \implies y'(t) = \tfrac{d}{dt}[\sin t] = \cos t \implies dy = y'(t)\ dt = \cos t\ dt$$

Rewrite line integral as an ordinary single integral & compute as appropriate:

$$\begin{aligned}
\int_C \Big(-xy^2\ dx + x^2\ dy\Big) &= \int_{C_1} \Big(-xy^2\ dx + x^2\ dy\Big) + \int_{-C_2} \Big(-xy^2\ dx + x^2\ dy\Big) \\
&= \int_{C_1} \Big(-xy^2\ dx + x^2\ dy\Big) - \int_{C_2} \Big(-xy^2\ dx + x^2\ dy\Big) \\
&= \int_0^1 \left[-(t-1)(1)(dt) + (t-1)^2(0\ dt)\right] - \int_0^{\pi/2} \left[-(\cos t)(\sin t)^2(-\sin t\ dt) + (\cos t)^2(\cos t\ dt)\right] \\
&= \int_0^1 (1-t)\ dt - \int_0^{\pi/2} \sin^3 t \cos t\ dt - \int_0^{\pi/2} \cos^3 t\ dt
\end{aligned}$$

The 3^{rd} integral is a CalcII integral!!

Since the integrand only involves an **odd power** of **cosine**,

factor $\cos t$ & apply the Pythagorean Identity to what remains:

$$\begin{aligned}
&= \int_0^1 (1-t)\ dt - \int_0^{\pi/2} \sin^3 t \cos t\ dt - \int_0^{\pi/2} \cos^2 t \cos t\ dt \\
&= \int_0^1 (1-t)\ dt - \int_0^{\pi/2} \sin^3 t \cos t\ dt - \int_0^{\pi/2} (1-\sin^2 t) \cos t\ dt
\end{aligned}$$

$CV:$ Let $u = \sin t$, then $du = \cos t\ dt$, and $u(0) = \sin(0) = 0 \quad u(\pi/2) = \sin(\pi/2) = 1$

$$\begin{aligned}
&\overset{CV}{=} \int_0^1 (1-t)\ dt - \int_0^1 u^3\ du - \int_0^1 (1-u^2)\ du \\
&= \left[t - \tfrac{1}{2}t^2\right]_{t=0}^{t=1} - \left[\tfrac{1}{4}u^4\right]_{u=0}^{u=1} - \left[u - \tfrac{1}{3}u^3\right]_{u=0}^{u=1} \\
&\overset{FTC}{=} \left[\left((1) - \tfrac{1}{2}(1)^2\right) - 0\right] - \left[\tfrac{1}{4}(1)^4 - 0\right] - \left[\left((1) - \tfrac{1}{3}(1)^3\right) - 0\right] \\
&= \boxed{-\frac{5}{12}}
\end{aligned}$$