EX 13.2.6: Find the work done by the force field $\vec{\mathbf{F}}(x,y,z) = \langle 2xy, x^2+2, y\rangle$ on an object moving along Γ, where Γ is the line segment from $(1,0,2)$ to $(3,4,1)$.

Recall the line integral formula for work done by a **force field** (instead of a constant force vector):

$$\text{Work} = \int_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$$

Parameterize Γ & encapsulate it into vector function $\vec{\mathbf{R}}(t)$:

$$\Gamma : \begin{cases} x(t) = 1 + (3-1)t = 2t+1 \\ y(t) = 0 + (4-0)t = 4t \\ z(t) = 2 + (1-2)t = 2 - t \\ t \in [0,1] \end{cases} \implies \vec{\mathbf{R}}(t) := \langle 2t+1, 4t, 2-t\rangle \text{ for } 0 \le t \le 1$$

Express differential $d\vec{\mathbf{R}}$ in terms of t & dt:

$$d\vec{\mathbf{R}} = \vec{\mathbf{R}}'(t)\ dt = \left\langle \frac{d}{dt}[2t+1], \frac{d}{dt}[4t], \frac{d}{dt}[2-t] \right\rangle dt = \langle 2,4,-1\rangle\ dt$$

Rewrite line integral as an equivalent ordinary single integral & compute it:

$$\begin{aligned}
\text{Work} &= \int_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} \\
&= \int_\Gamma \left\langle 2xy, x^2+2, y\right\rangle \cdot d\vec{\mathbf{R}} \\
&= \int_{\text{Starting } t\text{-value of } \Gamma}^{\text{Ending } t\text{-value of } \Gamma} \left\langle 2(2t+1)(4t), (2t+1)^2+2, (4t)\right\rangle \cdot \langle 2,4,-1\rangle\ dt \\
&= \int_0^1 \left\langle 16t^2+8t, 4t^2+4t+3, 4t\right\rangle \cdot \langle 2,4,-1\rangle\ dt \\
&= \int_0^1 \left[2\left(16t^2+8t\right) + 4\left(4t^2+4t+3\right) + (-1)(4t)\right]\ dt \\
&= \int_0^1 \left[32t^2+16t+16t^2+16t+12-4t\right]\ dt \\
&= \int_0^1 \left[48t^2+28t+12\right]\ dt \\
&= \left[16t^3+14t^2+12t\right]_{t=0}^{t=1} \\
&\overset{FTC}{=} \left[16(1)^3+14(1)^2+12(1)\right] - \left[16(0)^3+14(0)^2+12(0)\right] \\
&= \boxed{42}
\end{aligned}$$