**EX 13.3.1:** Let vector field $\vec{\mathbf{F}}(x,y) = \langle y^2, 2xy-3\rangle$ and $C$ be any smooth path from $P(0,0)$ to $Q(2,-1)$.

(a) Verify that $\vec{\mathbf{F}}$ is conservative. (b) Find a scalar potential $f$ for $\vec{\mathbf{F}}$. (c) Compute $I = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$.

(a) Let $M, N$ be the components of $\vec{\mathbf{F}}$: $M(x,y) = y^2$ $\qquad N(x,y) = 2xy$

Since vector field is 2D, compute the **cross partials**:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y^2\right] = 2y \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[2xy\right] = 2y$$

$\therefore$ Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\vec{\mathbf{F}}$ is conservative.

(b) A scalar potential $f$ can be found using either the 1-phi or 2-phi method.

**1-phi Method:**

$$\vec{\mathbf{F}} = \nabla f \implies \langle y^2,\ 2xy-3\rangle = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y}\right\rangle$$

Equate **one set** of components of $\vec{\mathbf{F}}$ & $\nabla f$: (here, the $2^{nd}$ components are equated)

$$\implies \frac{\partial f}{\partial y} = 2xy - 3 \implies f(x,y) = \int (2xy-3)\ dy = (\text{Treat } x \text{ as constant}) = xy^2 - 3y + \varphi(x)$$

$\therefore\ f(x,y) = xy^2 - 3y + \varphi(x)$

$$\implies \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xy^2 - 3y + \varphi(x)\right] = y^2 - 0 + \varphi'(x)$$

But $\dfrac{\partial f}{\partial x} = y^2$ since $\nabla f = \vec{\mathbf{F}}$

$\therefore\ y^2 + \varphi'(x) = y^2 \implies \varphi'(x) = 0 \implies \varphi(x) = \int 0\ dx = K,$ where $K \in \mathbb{R}$

$\therefore\ f(x,y) = xy^2 - 3y + \varphi(x) = xy^2 - 3y + K$

But only one scalar potential is needed, so let $K = 0$.

$\therefore$ Scalar Potential $\boxed{f(x,y) = xy^2 - 3y}$

**2-phi Method:**

$$\vec{\mathbf{F}} = \nabla f \implies \langle y^2,\ 2xy-3\rangle = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y}\right\rangle$$

Equate **both sets** of components of $\vec{\mathbf{F}}$ & $\nabla f$:

$$\frac{\partial f}{\partial x} = y^2 \implies f(x,y) = \int y^2\ dx = xy^2 + \varphi_1(y)$$

$$\frac{\partial f}{\partial y} = 2xy - 3 \implies f(x,y) = \int (2xy-3)\ dy = xy^2 - 3y + \varphi_2(x)$$

$\therefore\ f(x,y) = xy^2 + \varphi_1(y) = xy^2 - 3y + \varphi_2(x)$

$\implies xy^2 + \varphi_1(y) = xy^2 - 3y + \varphi_2(x)$

$\implies \varphi_1(y) = -3y + \varphi_2(x)$

$\implies \varphi_1(y) = -3y$ and $\varphi_2(x) = 0$ (Do you see why??)

$\therefore$ Scalar Potential $\boxed{f(x,y) = xy^2 - 3y}$

(c) Since $\vec{\mathbf{F}}$ is **conservative** (AKA a **gradient field**), use Fundamental Theorem for Line Integrals (FTLI):

$$I = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = \int_C \nabla f \cdot d\vec{\mathbf{R}} \overset{FTLI}{=} f(Q) - f(P) = f(2,-1) - f(0,0) = \left[(2)(-1)^2 - 3(-1)\right] - \left[(0)(0)^2 - 3(0)\right] = \boxed{5}$$