EX 13.3.3: Let C be the path traced by $\vec{\mathbf{R}}(t) = \langle 2\sin\left(\frac{\pi t}{2}\right)\cos(\pi t), \arcsin t\rangle$ for $0 \le t \le 1$.

(a) Verify that line integral $I = \displaystyle\int_C \Big[(\sin y)\ dx + (3 + x\cos y)\ dy\Big]$ is independent of path (IoP).

(b) Compute line integral I **without finding a scalar potential**.

(a) Let $\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle$, where $M(x,y) = \sin y$ and $N(x,y) = 3 + x\cos y$

Compute the cross partials: $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\Big[\sin y\Big] = \cos y$ $\quad \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\Big[3 + x\cos y\Big] = \cos y$

$\therefore$ Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, vector field $\vec{\mathbf{F}}$ is conservative $\implies$ Line integral I is independent of path

(b) Using the given path parameterization $\vec{\mathbf{R}}(t)$ results in a tedious single integral to compute!

So instead, pick a simpler path parameterization $\vec{\mathbf{R}}^*(t)$:

Find the starting & ending points of path C:

$\vec{\mathbf{R}}(0) = \left\langle 2\sin\left(\frac{\pi(0)}{2}\right)\cos(\pi(0)), \arcsin(0)\right\rangle = \langle 0, 0\rangle \implies$ Starting Point $P(0,0)$

$\vec{\mathbf{R}}(1) = \left\langle 2\sin\left(\frac{\pi(1)}{2}\right)\cos(\pi(1)), \arcsin(1)\right\rangle = \langle -2, \pi/2\rangle \implies$ Starting Point $Q(-2,\pi/2)$

The simplest path C^* from point P to point Q is a **line segment**:

Let $\vec{\mathbf{R}}^*(t) = \langle 0 + (-2-0)t, 0 + (\pi/2 - 0)t\rangle = \left\langle -2t, \frac{\pi}{2}t\right\rangle$ for $t \in [0,1]$

Then $d\vec{\mathbf{R}}^*(t) = \left\langle \frac{d}{dt}\left[-2t\right], \frac{d}{dt}\left[\frac{\pi}{2}t\right]\right\rangle\ dt = \left\langle -2, \frac{\pi}{2}\right\rangle\ dt$

Rewrite original line integral along path C as a line integral along path C^*, then rewrite as a single integral:

$$\begin{aligned}
I &= \int_C \Big[(\sin y)\ dx + (3 + x\cos y)\ dy\Big] \\
&= \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{R}} \\
&\overset{IoP}{=} \int_{C^*} \vec{\mathbf{F}}\cdot d\vec{\mathbf{R}}^* \\
&= \int_0^1 \left\langle \sin\left(\frac{\pi}{2}t\right), 3 + (-2t)\cos\left(\frac{\pi}{2}t\right)\right\rangle \cdot \left\langle -2, \frac{\pi}{2}\right\rangle\ dt \\
&= \int_0^1 \left[-2\sin\left(\frac{\pi}{2}t\right) + \frac{3\pi}{2} - \pi t\cos\left(\frac{\pi}{2}t\right)\right]\ dt \\
&= -2\int_0^1 \sin\left(\frac{\pi}{2}t\right)\ dt + \int_0^1 \frac{3\pi}{2}\ dt - \pi\int_0^1 t\cos\left(\frac{\pi}{2}t\right)\ dt
\end{aligned}$$

IBP: Let $\begin{cases} u = t \\ dv = \cos\left(\frac{\pi}{2}t\right)\ dt \end{cases} \implies \begin{cases} du = dt \\ v = \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) \end{cases}$

$$\begin{aligned}
&\overset{IBP}{=} -2\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}t\right)\right]_{t=0}^{t=1} + \left[\frac{3\pi}{2}t\right]_{t=0}^{t=1} - \pi\left[\left[\frac{2}{\pi}t\sin\left(\frac{\pi}{2}t\right)\right]_{t=0}^{t=1} - \int_0^1 \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\ dt\right] \\
&\overset{FTC}{=} -2\left(-\frac{2}{\pi}\right)[0-1] + \frac{3\pi}{2} - \pi\left[\frac{2}{\pi} - 0\right] + \pi\left[-\left(\frac{2}{\pi}\right)^2\cos\left(\frac{\pi}{2}t\right)\right]_{t=0}^{t=1} \\
&\overset{FTC}{=} -\frac{4}{\pi} + \frac{3\pi}{2} - 2 + \pi\left[0 + \frac{4}{\pi^2}\right] \\
&= \boxed{\frac{3\pi}{2} - 2}
\end{aligned}$$

REMARK: Here are two alternative parameterizations C^* involving a horizontal & vertical line segment: $(0 \le t \le 1)$

1. $C^* = C^*_{HL} \cup C^*_{VL}$, where $\vec{\mathbf{R}}^*_{HL}(t) = \langle 0 + (-2-0)t, 0\rangle = \langle -2t, 0\rangle$ and $\vec{\mathbf{R}}^*_{VL}(t) = \left\langle -2, 0 + \left(\frac{\pi}{2} - 0\right)t\right\rangle = \left\langle -2, \frac{\pi}{2}t\right\rangle$

—- OR —-

2. $C^* = C^*_{VL} \cup C^*_{HL}$, where $\vec{\mathbf{R}}^*_{VL}(t) = \left\langle 0, 0 + \left(\frac{\pi}{2} - 0\right)t\right\rangle = \left\langle 0, \frac{\pi}{2}t\right\rangle$ and $\vec{\mathbf{R}}^*_{HL}(t) = \left\langle 0 + (-2-0)t, \frac{\pi}{2}\right\rangle = \left\langle -2t, \frac{\pi}{2}\right\rangle$