EX 13.6.2: Let curve C be the intersection of plane $y+z=2$ & cylinder $x^2+y^2=1$ oriented CCW as viewed from above.

Use Stokes' Theorem to compute the line integral $I = \oint_C \left(-y^2\,dx + x\,dy + z^2\,dz\right)$.

In anticipation of Stokes' Theorem, let $\vec{\mathbf{F}}(x,y,z) = \left\langle -y^2, x, z^2 \right\rangle$ and $d\vec{\mathbf{R}} = \langle dx, dy, dz\rangle$

$$\text{Then } \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & x & z^2 \end{vmatrix} = \left\langle \tfrac{\partial}{\partial x}[z^2] - \tfrac{\partial}{\partial z}[x], \tfrac{\partial}{\partial z}[-y^2] - \tfrac{\partial}{\partial x}[z^2], \tfrac{\partial}{\partial x}[x] - \tfrac{\partial}{\partial y}[-y^2] \right\rangle = \langle 0, 0, 1+2y\rangle$$

Choose an appropriate surface S with C as it's boundary curve:

Since C is the intersection of two surfaces, S should be one of the two surfaces.

Pick the plane for S since it can be represented in the form $z = f(x,y)$.

Let S be the portion of plane $y+z=2$ inside cylinder $x^2+y^2=1$. Then, $y+z=2 \implies z = 2-y \implies f(x,y) = 2-y$

$$\implies f_x = \frac{\partial}{\partial x}\left[2-y\right] = 0, \qquad f_y = \frac{\partial}{\partial y}\left[2-y\right] = -1$$

Determine the **compatible** unit normal $\hat{\mathbf{N}}$ to the surface S and express $\hat{\mathbf{N}}\,dS$ in terms of dA:

Since path C is oriented CCW, unit normal $\hat{\mathbf{N}}$ should be upward by the Right-Hand Rule.

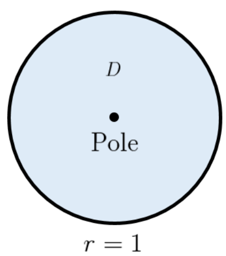
Since unit normal $\hat{\mathbf{N}}$ is upward, $\hat{\mathbf{N}}\,dS = \langle -f_x, -f_y, 1\rangle\,dA$ **(instead of $\hat{\mathbf{N}}\,dS = \langle f_x, f_y, -1\rangle\,dA$)**

$$\implies \hat{\mathbf{N}}\,dS = \langle -(0), -(-1), 1\rangle\,dA = \langle 0,1,1\rangle\,dA$$

Project S onto the region D on the xy-plane, sketch D and label BP's & BC's:

Since S is inside cylinder $x^2+y^2=1$ which projects onto circle $x^2+y^2=1$ on xy-plane, D is closed disk $x^2+y^2 \le 1$.

Since D is a **disk**, expect to write eventual double integral in **polar coordinates**.



Use Stokes' Thm to write line integral as flux integral, then write flux integral as double integral, finally compute it:

$$\begin{aligned} I &= \oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} \\ &\overset{STOKES}{=} \iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \hat{\mathbf{N}}\,dS \\ &= \iint_D \langle 0,0,1+2y\rangle \cdot \langle 0,1,1\rangle\,dA \\ &= \iint_D (1+2y)\,dA \\ &\overset{POLAR}{=} \int_{\text{Smallest }\theta\text{-value}}^{\text{Largest }\theta\text{-value}} \int_{\text{Pole}}^{\text{Outer BC}} [1+2(r\sin\theta)]\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(r + 2r^2\sin\theta\right)\,dr\,d\theta \\ &= \text{(I assume you're comfortable with computing iterated double integrals)} = \boxed{\pi} \end{aligned}$$