EX 13.7.2: Let solid E be bounded by the planes $z = 0$, $y = 0$, $y = 2$, and the parabolic cylinder $z = 1 - x^2$.

Let surface S be the boundary of solid E with outward unit normal $\widehat{\mathbf{N}}$.

Let vector field $\vec{\mathbf{F}}(x, y, z) = \langle x + \arctan y, 2y + e^z, z^2 - \sin x \rangle$.

Use the Divergence Theorem (AKA Gauss' Theorem) to compute the surface integral $I = \oiint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS$.

In anticipation of using Gauss' Theorem, compute the **divergence** of $\vec{\mathbf{F}}$:

$$\nabla \cdot \vec{\mathbf{F}} = \frac{\partial}{\partial x}\Big[x + \arctan y\Big] + \frac{\partial}{\partial y}\Big[2y + e^z\Big] + \frac{\partial}{\partial z}\Big[z^2 - \sin x\Big] = (1+0) + (2+0) + (2z - 0) = 3 + 2z$$

$$\therefore \quad \nabla \cdot \vec{\mathbf{F}} = 3 + 2z$$

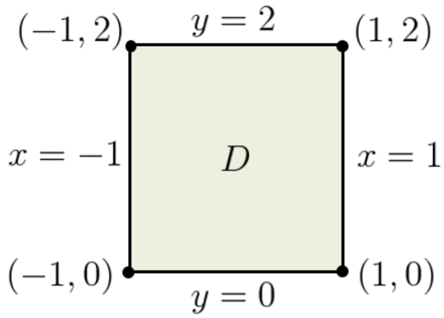
Since solid E is not easily described in spherical coordinates, project solid E onto region D in xy-plane:

BS's of E, planes $y = 0$ & $y = 2$, are trivially projected onto lines $y = 0$ & $y = 2 \leftarrow$ Top & Bottom BC's of D

Intersect top BS (parabolic cylinder) with bottom BS (plane $z = 0$):

$0 = 1 - x^2 \implies$ lines $x = -1$ & $x = 1 \leftarrow$ Left & Right BC's of D

Since D is a **square**, expect to write eventual iterated triple integral in **rectangular coordinates**.



Use Gauss' Thm to write closed flux integral as triple integral, then write it as iterated triple integral, finally compute it:

$$\begin{aligned}
I &= \oiint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS \\
&\overset{GAUSS}{=} \iiint_E \nabla \cdot \vec{\mathbf{F}}\, dV \\
&= \iiint_E (3 + 2z)\, dV \\
&= \iint_D \int_{\text{Btm BS of } E}^{\text{Top BS of } E} (3 + 2z)\, dz\, dA \\
&= \int_{\text{Smallest } x\text{-coord in } D}^{\text{Largest } x\text{-coord in } D} \int_{\text{Btm BC of } D}^{\text{Top BC of } D} \int_{\text{Btm BS of } E}^{\text{Top BS of } E} (3 + 2z)\, dz\, dy\, dx \\
&= \int_{-1}^{1} \int_0^2 \int_0^{1 - x^2} (3 + 2z)\, dz\, dy\, dx \\
&= \text{(I assume you're comfortable with computing iterated triple integrals)} = \boxed{\frac{152}{15}}
\end{aligned}$$