EX 9.7.13: Given the quadric surface $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0$,

(a) Find the intersection of the quadric surface with the plane $z = 8$

Plug plane into quadric surface & simplify:

$$z = 8 \implies \frac{x^2}{4} + \frac{y^2}{9} - \frac{(8)^2}{16} = 0 \implies \implies \frac{x^2}{4} + \frac{y^2}{9} = 4 \implies \boxed{\frac{x^2}{16} + \frac{y^2}{36} = 1 \longleftarrow \text{ Ellipse}}$$

(b) Find the intersection of the quadric surface with the plane $z = 0$

Plug plane into quadric surface & simplify:

$$z = 0 \implies \frac{x^2}{4} + \frac{y^2}{9} - \frac{(0)^2}{16} = 0 \implies \implies \frac{x^2}{4} + \frac{y^2}{9} = 0 \implies \begin{cases} x = 0 \\ y = 0 \end{cases} \implies \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases} \implies \boxed{(0,0,0) \longleftarrow \text{ Origin}}$$

(c) Find the intersection of the quadric surface with the plane $y = -3$

Plug plane into quadric surface & simplify:

$$y = -3 \implies \frac{x^2}{4} + \frac{(-3)^2}{9} - \frac{z^2}{16} = 0 \implies \implies \frac{x^2}{4} - \frac{z^2}{16} = -1 \implies \boxed{\frac{z^2}{16} - \frac{x^2}{4} = 1 \longleftarrow \text{ Hyperbola}}$$

(d) Find the intersection of the quadric surface with the plane $y = 0$

Plug plane into quadric surface & simplify:

$$y = 0 \implies \frac{x^2}{4} + \frac{(0)^2}{9} - \frac{z^2}{16} = 0 \implies \implies \frac{x^2}{4} - \frac{z^2}{16} = 0$$

Now, $\frac{x^2}{4} - \frac{z^2}{16} = 0$ has at least one solution but is not a proper conic section since RHS is zero.

Therefore, solve for one of the variables, say x:

$$\frac{x^2}{4} - \frac{z^2}{16} = 0 \implies \frac{x^2}{4} = \frac{z^2}{16} \implies \pm\frac{x}{2} = \pm\frac{z}{4} \implies \frac{x}{2} = \mp\frac{z}{4} \overset{REMARK\#2}{\implies} \frac{x}{2} = \pm\frac{z}{4} \implies x = \pm\frac{z}{2} \implies x = \frac{1}{2}z, x = -\frac{1}{2}z$$

Therefore, $\boxed{\text{Intersection is the two lines } x = \frac{1}{2}z, x = -\frac{1}{2}z \text{ in the } xz\text{-plane which intersect at the origin}}$

(e) Find the intersection of the quadric surface with the plane $x = 0$

Plug plane into quadric surface & simplify:

$$x = 0 \implies \frac{(0)^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0 \implies \frac{y^2}{9} - \frac{z^2}{16} = 0$$

Now, $\frac{y^2}{9} - \frac{z^2}{16} = 0$ has at least one solution but is not a proper conic section since RHS is zero.

Therefore, solve for one of the variables, say y:

$$\frac{y^2}{9} - \frac{z^2}{16} = 0 \implies \frac{y^2}{9} = \frac{z^2}{16} \implies \pm\frac{y}{3} = \pm\frac{z}{4} \implies \frac{y}{3} = \mp\frac{z}{4} \overset{REMARK\#2}{\implies} \frac{y}{3} = \pm\frac{z}{4} \implies y = \pm\frac{3}{4}z \implies y = \frac{3}{4}z, y = -\frac{3}{4}z$$

Therefore, $\boxed{\text{Intersection is the two lines } y = \frac{3}{4}z, y = -\frac{3}{4}z \text{ in the } yz\text{-plane which intersect at the origin}}$

(f) Find the intersection of the quadric surface with the plane $x = -2$

$$x = -2 \implies \frac{(-2)^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0 \implies 1 + \frac{y^2}{9} - \frac{z^2}{16} = 0 \implies \frac{y^2}{9} - \frac{z^2}{16} = -1$$

Now, $\frac{y^2}{9} - \frac{z^2}{16} = -1$ has at least one solution & RHS $\neq 0$, so produce canonical form:

$$\frac{y^2}{9} - \frac{z^2}{16} = -1 \implies \boxed{\frac{z^2}{16} - \frac{y^2}{9} = 1 \longleftarrow \text{ Hyperbola}}$$

REMARK #1: RHS $\equiv$ "Right-Hand Side (of equation)"

REMARK #2: $w = \pm 7 \implies w \in \{-7, 7\}$. $w = \mp 7 \implies w \in \{-7, 7\}$. Thus, $\pm 7 = \mp 7$