VECTOR FUNCTIONS: ALGEBRA & LIMITS [SST 10.1]

- **THE FUNCTION LANDSCAPE (SO FAR):**

FUNCTION TYPE	PROTOTYPE	MAPPING	MEANING	FIRST SEEN
Scalar Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$	f maps scalar $\to$ scalar	Algebra
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$	$\mathbf{F}$ maps scalar $\to$ 2D vector	Calc III (Ch10)
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$	$\mathbf{F}$ maps scalar $\to$ 3D vector	Calc III (Ch10)

- **VECTOR FUNCTIONS (DEFINITION):**
  - 2D Vector Function $\mathbf{F}(t) := \langle f_1(t), f_2(t) \rangle = f_1(t)\widehat{\mathbf{i}} + f_2(t)\widehat{\mathbf{j}}$, where f_1, f_2 are scalar functions.
    - In other words: $\boxed{\text{Input: Scalar } t} \to \boxed{\mathbf{F}(t)} \to \boxed{\text{Output: 2D Vector } \langle f_1(t), f_2(t) \rangle}$
    - $\text{Dom}(\mathbf{F}) := \text{Dom}(f_1) \cap \text{Dom}(f_2)$
  - 3D Vector Function $\mathbf{F}(t) := \langle f_1(t), f_2(t), f_3(t) \rangle = f_1(t)\widehat{\mathbf{i}} + f_2(t)\widehat{\mathbf{j}} + f_3(t)\widehat{\mathbf{k}}$, where f_1, f_2, f_3 are scalar functions.
    - In other words: $\boxed{\text{Input: Scalar } t} \to \boxed{\mathbf{F}(t)} \to \boxed{\text{Output: 3D Vector } \langle f_1(t), f_2(t), f_3(t) \rangle}$
    - $\text{Dom}(\mathbf{F}) := \text{Dom}(f_1) \cap \text{Dom}(f_2) \cap \text{Dom}(f_3)$
  - Think of a vector function as a "vector encapsulation" of a **parametric equation**. Recall, t is called the **parameter**.
  - If the **initial point** of vector function $\mathbf{F}(t)$ is chosen to be the **origin** for all t, then $\mathbf{F}(t)$ is called a **position vector**.
    - As t varies, **position vector** $\mathbf{F}(t)$ traces out a **curve** that is denoted Γ.
  - REMARK: The focus will be on 3D vector functions. Remove 3^{rd} component terms to get 2D vector function result.

- **VECTOR FUNCTIONS (ALGEBRA):**
  - Assume $\mathbf{F}(t), \mathbf{G}(t), \mathbf{H}(t)$ are vector functions & $f(t), h(t)$ are scalar functions.

OPERATION	FORMULA	RESULT	REMARKS
Addition/Subtraction	$\mathbf{H}(t) = \mathbf{F}(t) \pm \mathbf{G}(t)$	Vector Function	$\text{Dom}(\mathbf{H}) = \text{Dom}(\mathbf{F}) \cap \text{Dom}(\mathbf{G})$
Scalar Multiplication	$\mathbf{H}(t) = f(t)\mathbf{F}(t)$	Vector Function	$\text{Dom}(\mathbf{H}) = \text{Dom}(f) \cap \text{Dom}(\mathbf{F})$
Dot Product	$h(t) = \mathbf{F}(t) \cdot \mathbf{G}(t)$	**Scalar Function**	$\text{Dom}(h) = \text{Dom}(\mathbf{F}) \cap \text{Dom}(\mathbf{G})$
Cross Product	$\mathbf{H}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$	Vector Function	$\text{Dom}(\mathbf{H}) = \text{Dom}(\mathbf{F}) \cap \text{Dom}(\mathbf{G})$ $\mathbf{G}(t) \times \mathbf{F}(t) = -\mathbf{F}(t) \times \mathbf{G}(t)$

- **VECTOR FUNCTIONS (LIMITS):**
  - Suppose $\lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \lim_{t \to t_0} f_3(t)$ are all finite. Then: $\lim_{t \to t_0} \mathbf{F}(t) := \left[\lim_{t \to t_0} f_1(t)\right]\widehat{\mathbf{i}} + \left[\lim_{t \to t_0} f_2(t)\right]\widehat{\mathbf{j}} + \left[\lim_{t \to t_0} f_3(t)\right]\widehat{\mathbf{k}}$
  - Otherwise, if any of the above three scalar limits are $-\infty, +\infty$ or DNE, then $\lim_{t \to t_0} \mathbf{F}(t) = \text{DNE}$.

LIMIT RULE	FORMULA
Sum/Diff Rule	$\lim_{t \to t_0} [\mathbf{F}(t) \pm \mathbf{G}(t)] = \left[\lim_{t \to t_0} \mathbf{F}(t)\right] \pm \left[\lim_{t \to t_0} \mathbf{G}(t)\right]$
Scalar Multiple Rule	$\lim_{t \to t_0} [f(t)\mathbf{F}(t)] = \left[\lim_{t \to t_0} f(t)\right] \left[\lim_{t \to t_0} \mathbf{F}(t)\right]$
Dot Product Rule	$\lim_{t \to t_0} [\mathbf{F}(t) \cdot \mathbf{G}(t)] = \left[\lim_{t \to t_0} \mathbf{F}(t)\right] \cdot \left[\lim_{t \to t_0} \mathbf{G}(t)\right]$
Cross Product Rule	$\lim_{t \to t_0} [\mathbf{F}(t) \times \mathbf{G}(t)] = \left[\lim_{t \to t_0} \mathbf{F}(t)\right] \times \left[\lim_{t \to t_0} \mathbf{G}(t)\right]$

- **VECTOR FUNCTIONS (CONTINUITY):**
  - Vector function $\mathbf{F}(t)$ is **continuous at point** $t_0 \iff \mathbf{F} \in C(\{t_0\}) \iff \left(t_0 \in \text{Dom}(\mathbf{F}) \text{ AND } \lim_{t \to t_0} \mathbf{F}(t) = \mathbf{F}(t_0)\right)$
  - Vector function $\mathbf{F}(t)$ is **continuous on a set** $S \iff \mathbf{F} \in C(S) \iff$ scalar functions $f_1, f_2, f_3 \in C(S)$

EX 10.1.1: Encapsulate the parametric equation into the position vector $\mathbf{R}(t)$: $\begin{cases} x = t^2 \\ y = e^{3t} \\ z = 1 - \cos 4t \\ t \in \mathbb{R} \end{cases}$

EX 10.1.2: Find the position vector $\mathbf{R}(t)$ that traces the curve Γ,

where Γ is the intersection of the plane $x + y + z - 2 = 0$ with the parabolic cylinder $z = 3x^2$.

EX 10.1.3: Find the position vector $\mathbf{R}(t)$ that traces the curve Γ,

where Γ is the intersection of the planes $\mathbb{P}_1 : x + y + z - 2 = 0$ and $\mathbb{P}_2 : 2x - y + 3z + 4 = 0$.

EX 10.1.4: Find the position vector $\mathbf{R}(t)$ that traces the curve Γ,

where Γ is the intersection of the elliptic paraboloid $z = x^2 + y^2$ with the elliptic cylinder $\dfrac{x^2}{2} + \dfrac{y^2}{9} = 1$.

EX 10.1.5: Find the domain of the vector function $\mathbf{F}(t) = \left\langle e^t, 5t - 7, \sin t \cos(3t) \right\rangle$

EX 10.1.6: Find the domain of the vector function $\mathbf{F}(t) = \sqrt{t+6}\,\widehat{\mathbf{i}} - \left(\dfrac{19}{t-100}\right)\widehat{\mathbf{j}} + (\ln t)\,\widehat{\mathbf{k}}$

EX 10.1.7: Let $\mathbf{F}(t) = \left\langle 1, t, t^2 \right\rangle$ and $\mathbf{G}(t) = \left(e^t\right)\widehat{\mathbf{i}} - (\cos t)\widehat{\mathbf{k}}$.

Compute: (a) $3\mathbf{F}(t) - 2t^2\mathbf{G}(t)$ (b) $\mathbf{F}(t) \cdot \mathbf{G}(t)$ (c) $\mathbf{F}(t) \times \mathbf{G}(t)$ (d) $\mathbf{G}(t) \times \mathbf{F}(t)$

EX 10.1.8: Let $\mathbf{F}(t) = \langle 1, t, t^2 \rangle$.

Evaluate limits: (a) $\lim_{t \to 2} \mathbf{F}(t)$ (b) $\lim_{t \to 2^-} \mathbf{F}(t)$ (c) $\lim_{t \to 2^+} \mathbf{F}(t)$

EX 10.1.9: Let $\mathbf{G}(t) = \left(t^{-2}\right) \widehat{\mathbf{i}} + \left(\frac{1}{t}\right) \widehat{\mathbf{j}} - \left(e^{8t}\right) \widehat{\mathbf{k}}$.

Evaluate limits: (a) $\lim_{t \to -\infty} \mathbf{G}(t)$ (b) $\lim_{t \to \infty} \mathbf{G}(t)$ (c) $\lim_{t \to 0^-} \mathbf{G}(t)$ (d) $\lim_{t \to 0^+} \mathbf{G}(t)$ (e) $\lim_{t \to 0} \mathbf{G}(t)$