VECTOR FUNCTIONS: CALCULUS & KINEMATICS [SST 10.2]

- **THE FUNCTION LANDSCAPE (SO FAR):**

FUNCTION TYPE	PROTOTYPE	MAPPING	MEANING	FIRST SEEN
Scalar Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$	f maps scalar $\to$ scalar	Algebra
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$	$\mathbf{F}$ maps scalar $\to$ 3D vector	Calc III (Ch10)

- **VECTOR FUNCTIONS (DERIVATIVES):**
  - LIMIT DEFINITION OF DERIVATIVE: $\mathbf{F}'(t) := \lim\limits_{\Delta t \to 0} \dfrac{\mathbf{F}(t+\Delta t) - \mathbf{F}(t)}{\Delta t}$

ORDER	DEFINITION	LEIBNIZ NOTATION
1^{st} Derivative	$\mathbf{F}'(t) := \langle f_1'(t), f_2'(t), f_3'(t) \rangle$	$\dfrac{d\mathbf{F}}{dt}$
2^{nd} Derivative	$\mathbf{F}''(t) := \langle f_1''(t), f_2''(t), f_3''(t) \rangle$	$\dfrac{d^2\mathbf{F}}{dt^2}$
n^{th} Derivative	$\mathbf{F}^{(n)}(t) := \left\langle f_1^{(n)}(t), f_2^{(n)}(t), f_3^{(n)}(t) \right\rangle$	$\dfrac{d^n\mathbf{F}}{dt^n}$

  - Vector function $\mathbf{F}(t)$ is **differentiable on a set** $S \subseteq \mathbb{R} \iff \mathbf{F} \in C^1(S) \iff$ scalar functions $f_1, f_2, f_3 \in C^1(S)$
  - Vector function $\mathbf{F}(t)$ is **twice differentiable on a set** $S \subseteq \mathbb{R} \iff \mathbf{F} \in C^2(S) \iff f_1, f_2, f_3 \in C^2(S)$

DERIVATIVE RULE	FORMULA
Sum/Diff Rule	$\dfrac{d}{dt}\Big[\mathbf{F}(t) \pm \mathbf{G}(t)\Big] = \mathbf{F}'(t) \pm \mathbf{G}'(t)$
Scalar Multiple Rule	$\dfrac{d}{dt}\Big[h(t)\mathbf{F}(t)\Big] = h'(t)\mathbf{F}(t) + h(t)\mathbf{F}'(t)$
Dot Product Rule	$\dfrac{d}{dt}\Big[\mathbf{F}(t) \cdot \mathbf{G}(t)\Big] = \mathbf{F}'(t) \cdot \mathbf{G}(t) + \mathbf{F}(t) \cdot \mathbf{G}'(t)$
Cross Product Rule	$\dfrac{d}{dt}\Big[\mathbf{F}(t) \times \mathbf{G}(t)\Big] = \mathbf{F}'(t) \times \mathbf{G}(t) + \mathbf{F}(t) \times \mathbf{G}'(t)$
Chain Rule	$\dfrac{d}{dt}\Big[\mathbf{F}\left[h(t)\right]\Big] = \mathbf{F}'\left[h(t)\right] h'(t)$

- **SMOOTH CURVES:** Let Γ be the curve traced by **position vector** $\mathbf{F}(t)$.
  - $\mathbf{F}(t)$ (curve Γ) is **smooth on interval** $(a,b) \iff \Big[\mathbf{F} \in C^1(a,b) \text{ AND } \mathbf{F}'(t) \neq \vec{\mathbf{0}} \quad \forall t \in (a,b)\Big]$
  - $\mathbf{F}(t)$ (curve Γ) is **piecewise smooth on** $(a,b) \iff \Gamma$ is smooth on (a,b) except at a finite # of points.

- **VECTOR FUNCTIONS (KINEMATICS):**
  - SETUP: Given a **particle** in 3D with **position vector** $\mathbf{R}(t) = \langle R_1(t), R_2(t), R_3(t) \rangle$

Trajectory			$\Gamma :=$ Graph of position vector $\mathbf{R}(t)$		
Velocity	Speed	Direction	$\mathbf{V}(t) := \mathbf{R}'(t)$	$\lvert\lvert\mathbf{V}(t)\rvert\rvert$	$\dfrac{\mathbf{V}(t)}{\lvert\lvert\mathbf{V}(t)\rvert\rvert}$
Acceleration Vector			$\mathbf{A}(t) := \mathbf{V}'(t) = \mathbf{R}''(t)$		

  - The particle is **stationary** $\iff \mathbf{V}(t) = \vec{\mathbf{0}}$

- **VECTOR FUNCTIONS (INTEGRALS):**

$\star$ $\displaystyle\int \mathbf{F}(t)\,dt := \left[\int f_1(t)\,dt\right]\hat{\mathbf{i}} + \left[\int f_2(t)\,dt\right]\hat{\mathbf{j}} + \left[\int f_3(t)\,dt\right]\hat{\mathbf{k}} + \vec{\mathbf{C}}$, where $\vec{\mathbf{C}} = \langle C_1, C_2, C_3 \rangle$ is a **constant vector**

$\star$ $\displaystyle\int_a^b \mathbf{F}(t)\,dt := \left[\int_a^b f_1(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b f_2(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b f_3(t)\,dt\right]\hat{\mathbf{k}}$

EX 10.2.1: Let $\mathbf{F}(t) = \left\langle 1, t, t^5 \right\rangle$. Compute: (a) $\mathbf{F}'(t)$ (b) $\mathbf{F}''(t)$ (c) $\mathbf{F}'''(t)$ (d) $\mathbf{F}^{(13)}(t)$.

EX 10.2.2: Let $\mathbf{G}(t) = (\cos t)\widehat{\mathbf{i}} - (\sin(5t))\widehat{\mathbf{j}} + (\tan t)\widehat{\mathbf{k}}$. Compute: (a) $\dfrac{d\mathbf{G}}{dt}$ (b) $\dfrac{d^2\mathbf{G}}{dt^2}$.

EX 10.2.3: Let $f(t) = \left\langle 1, t, t^2 \right\rangle \cdot \left[\left(e^t\right)\widehat{\mathbf{i}} - \left(t^{-2}\right)\widehat{\mathbf{j}} + 3\widehat{\mathbf{k}} \right]$. Compute $f'(t)$.

EX 10.2.4: Let $g(\theta) = ||\langle \cos(3\theta), \sqrt{\theta + 1}, \sin(3\theta) \rangle||^2$. Compute $g'(\theta)$.

EX 10.2.5: Let $\mathbf{F}(t) = \langle 1, t, t^2 \rangle$ and $\mathbf{w} = \langle 3, -1, 1 \rangle$.

Compute: (a) $\dfrac{d}{dt}\left[3t\mathbf{F}(t) + t^2\mathbf{w}\right]$ (b) $\dfrac{d}{dt}\left[3t\mathbf{F}(t) \cdot t^2\mathbf{w}\right]$ (c) $\dfrac{d}{dt}\left[3t\mathbf{F}(t) \times t^2\mathbf{w}\right]$

EX 10.2.6: Let $\mathbf{R}(t) = \langle t^3 - 3t, t^4, 4t^2 \rangle$ be the position vector of a particle in space at time t.

At $t = 1$, find the particle's: (a) velocity (b) acceleration (c) speed (d) direction

(e) When, if ever, is the particle stationary?

EX 10.2.7: Let $\mathbf{F}(t) = (3t^2)\,\widehat{\mathbf{i}} - \left(\frac{7}{t}\right)\widehat{\mathbf{k}}$. Compute: (a) $\int \mathbf{F}(t)\ dt$ (b) $\int_e^{e^3} \mathbf{F}(t)\ dt$

EX 10.2.8: Let $\mathbf{G}(t) = e^t \left\langle 1, e^{t/2}, t \right\rangle$. Compute: (a) $\int \mathbf{G}(t)\ dt$ (b) $\int_0^1 \mathbf{G}(t)\ dt$

EX 10.2.9: Given a particle's velocity $\mathbf{V}(t) = \langle 1, t, t^2 \rangle$ and initial position $\mathbf{R}(0) = \langle 1, 2, 3 \rangle$, find its position vector, $\mathbf{R}(t)$.

EX 10.2.10: Given a particle's acceleration $\mathbf{A}(t) = \langle t, e^t, \sin t \rangle$, initial velocity $\mathbf{V}(0) = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, and initial position $\mathbf{R}(0) = \langle 1, -2, 2 \rangle$, find its: (a) velocity vector $\mathbf{V}(t)$ (b) position vector $\mathbf{R}(t)$.

EX 10.2.11: Is $\mathbf{F}(t) = \left\langle t^2, \cos t, \sin\left(t^2\right) \right\rangle$ smooth or piecewise smooth over interval $(-\pi, \pi)$? (Justify answer)

EX 10.2.12: Is $\mathbf{G}(t) = \left\langle e^t - t, t^{3/2}, \cos(\pi t) \right\rangle$ smooth or piecewise smooth over interval $(0, 1)$? (Justify answer)