VECTOR FUNCTIONS: TNB-FRAME, CURVATURE, TORSION [SST 10.4]

- **THE FUNCTION LANDSCAPE (SO FAR):**

FUNCTION TYPE	PROTOTYPE	MAPPING	MEANING	FIRST SEEN
Scalar Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$	f maps scalar $\to$ scalar	Algebra
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$	$\mathbf{F}$ maps scalar $\to$ 3D vector	Calc III (Ch10)

- **VECTOR FUNCTIONS (TNB-FRAME):**
  - Let **position vector** $\mathbf{R}(t) = \langle x(t), y(t), z(t)\rangle$ trace out a **smooth curve** Γ.

CONCEPT	FORMULA	REMARKS
Arc Length of Γ on interval $[t_1, t_2]$	$\displaystyle\int_{t_1}^{t_2} \|\|\mathbf{R}'(t)\|\| \, dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$	Also valid if Γ is only piecewise smooth
Unit Tangent (Vector)	$\widehat{\mathbf{T}}(t) := \dfrac{\mathbf{R}'(t)}{\|\|\mathbf{R}'(t)\|\|}$	
Unit Normal (Vector)	$\widehat{\mathbf{N}}(t) := \dfrac{\widehat{\mathbf{T}}'(t)}{\left\|\left\|\widehat{\mathbf{T}}'(t)\right\|\right\|}$	
Curvature of Γ at t	$\kappa(t) := \dfrac{\left\|\left\|\widehat{\mathbf{T}}'(t)\right\|\right\|}{\|\|\mathbf{R}'(t)\|\|}$	Measures curve's tendency to "bend"
Unit Binormal (Vector)	$\widehat{\mathbf{B}}(t) := \widehat{\mathbf{T}}(t) \times \widehat{\mathbf{N}}(t)$	
Torsion of Γ at t	$\tau(t) := -\widehat{\mathbf{N}}(t) \cdot \widehat{\mathbf{B}}'(t)$	Measures curve's tendency to "twist"

- **2D CURVES (CURVATURE):**

1. Encapsulate curve into a 3D position vector $\mathbf{R}(t)$:

CURVE TYPE	PROTOTYPE	3D POSITION VECTOR ENCAPSULATION
2D Parametric Curve	$\begin{cases} x = f(t) \\ y = g(t) \\ t \in I \end{cases}$	$\mathbf{R}(t) := \langle f(t), g(t), 0\rangle$
2D Cartesian Curve	$y = f(x)$	$\mathbf{R}(t) := \langle t, f(t), 0\rangle$
2D Cartesian Curve	$x = g(y)$	$\mathbf{R}(t) := \langle g(t), t, 0\rangle$
2D Polar Curve	$r = f(\theta)$	$\mathbf{R}(t) := \langle f(t)\cos t, f(t)\sin t, 0\rangle$

2. Compute $\mathbf{R}'(t)$
3. Compute Unit Tangent $\widehat{\mathbf{T}}(t)$
4. Compute $\widehat{\mathbf{T}}'(t)$
5. Compute Unit Normal $\widehat{\mathbf{N}}(t)$
6. Compute Curvature $\kappa(t)$
7. Compute Unit Binormal $\widehat{\mathbf{B}}(t)$
8. Compute $\widehat{\mathbf{B}}'(t)$
9. Compute Torsion $\tau(t)$

EX 10.4.1: Let position vector $\mathbf{R}(t) = \langle 3 - \sqrt{2}t, 2 - e^t, 1 + e^{-t} \rangle$ trace a curve Γ.

Compute the arc length of Γ over interval $t \in [0, 1]$.

EX 10.4.2: Let position vector $\mathbf{R}(t) = \langle t, t^2, t^3 \rangle$ trace a curve Γ.

Setup integral to compute the arc length of Γ over interval $t \in [2, 5]$.

EX 10.4.3: Encapsulate 2D parametric curve $\begin{cases} x = 3\cos t \\ y = 5\sin t \\ t \in \mathbb{R} \end{cases}$ into 3D position vector $\mathbf{R}(t)$.

EX 10.4.4: Encapsulate 2D Cartesian curve $y = x^2 e^x$ into 3D position vector $\mathbf{R}(t)$.

EX 10.4.5: Encapsulate 2D Cartesian curve $x = y \tan y$ into 3D position vector $\mathbf{R}(t)$.

EX 10.4.6: Encapsulate 2D polar curve $r = 3\theta$ into 3D position vector $\mathbf{R}(t)$.

EX 10.4.7: Let position vector $\mathbf{R}(t) = \left\langle \frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}} \right\rangle$ trace a curve Γ. Compute:

(a) Unit Tangent $\widehat{\mathbf{T}}(t)$ (b) Unit Normal $\widehat{\mathbf{N}}(t)$ (c) Curvature $\kappa(t)$ (d) Unit Binormal $\widehat{\mathbf{B}}(t)$ (e) Torsion $\tau(t)$

EX 10.4.8: Let position vector $\mathbf{R}(t) = \left\langle \frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t \right\rangle$ trace a curve Γ. Compute:

(a) Unit Tangent $\widehat{\mathbf{T}}(t)$ (b) Unit Normal $\widehat{\mathbf{N}}(t)$ (c) Curvature $\kappa(t)$ (d) Unit Binormal $\widehat{\mathbf{B}}(t)$ (e) Torsion $\tau(t)$