FUNCTIONS OF SEVERAL VARIABLES: INTRODUCTION [SST 11.1]

- **THE FUNCTION LANDSCAPE (SO FAR):**

FUNCTION TYPE	PROTOTYPE	MAPPING	GRAPH	FIRST SEEN
(Scalar) Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$	2D Curve in xy-plane	Algebra
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$	3D Curve in xyz-space	Calc III (Ch10)
(Scalar) Function of 2 Variables	$z = f(x, y)$	$f : \mathbb{R}^2 \to \mathbb{R}$	3D Surface in xyz-space	Calc III (Ch11)
(Scalar) Function of 3 Variables	$w = f(x, y, z)$	$f : \mathbb{R}^3 \to \mathbb{R}$	4D Hypersurface in $xyzw$-space	Calc III (Ch11)

- **FUNCTIONS OF ONE VARIABLE (DOMAIN):**
  - $f \in \{1, x, x^2, x^3, x^4, x^5, \dots\} \implies \text{Dom}(f) = \mathbb{R}$
  - $f \in \{\sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}, \sqrt[8]{x}, \dots\} \implies [\,\text{Dom}(f) = [0, \infty) \iff x \geq 0]$
  - $f \in \{\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}, \sqrt[9]{x}, \dots\} \implies \text{Dom}(f) = \mathbb{R}$
  - $f \in \left\{\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \frac{1}{x^5}, \frac{1}{x^6}, \cdots\right\} \implies [\,\text{Dom}(f) = \mathbb{R} \setminus \{0\} \iff x \neq 0]$
  - $f \in \left\{\frac{1}{\sqrt{x}}, \frac{1}{\sqrt[4]{x}}, \frac{1}{\sqrt[6]{x}}, \frac{1}{\sqrt[8]{x}}, \cdots\right\} \implies [\,\text{Dom}(f) = \mathbb{R}_+ = (0, \infty) \iff x > 0]$
  - $f \in \left\{\frac{1}{\sqrt[3]{x}}, \frac{1}{\sqrt[5]{x}}, \frac{1}{\sqrt[7]{x}}, \frac{1}{\sqrt[9]{x}}, \cdots\right\} \implies \text{Dom}(f) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$
  - $f \in \{e^x, (-2)^x, 3^x, (\sqrt{5})^x, \dots\} \implies \text{Dom}(f) = \mathbb{R}$
  - $f \in \{\ln x, \log x, \log_b x, \dots\} \implies \text{Dom}(f) = \mathbb{R}_+ = (0, \infty)$
  - $f \in \left\{\frac{1}{\ln x}, \frac{1}{\log x}, \frac{1}{\log_b x}, \cdots\right\} \implies \text{Dom}(f) = \mathbb{R}_+ \setminus \{1\} = (0, 1) \cup (1, \infty)$
  - $f \in \{|x|, \sin x, \cos x\} \implies \text{Dom}(f) = \mathbb{R}$
  - $f \in \{\tan x, \sec x\} \implies \left[\,\text{Dom}(f) = \mathbb{R} \setminus \{\cos x = 0\} \iff x \notin \left\{\cdots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\right\}\right]$
  - $f \in \{\cot x, \csc x\} \implies \left[\,\text{Dom}(f) = \mathbb{R} \setminus \{\sin x = 0\} \iff x \notin \{\dots, -4\pi, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, \dots\}\right]$
- **FUNCTIONS OF TWO VARIABLES (DOMAIN):**
  - $\text{Dom}\,[f(x, y)] = \{(x, y) \in \mathbb{R}^2 : f(x, y) \text{ is defined}\}$
  - $f(x, y) \in \{1, x, x^2, x^3, \dots, y, y^2, y^3, \dots, xy, xy^2, x^2y, x^2y^2, \dots\} \implies \text{Dom}\,[f(x, y)] = \mathbb{R}^2$
  - $\alpha \neq 0 \implies \text{Dom}(\alpha f) = \text{Dom}(f)$
  - $\text{Dom}(f \pm g) = \text{Dom}(f) \cap \text{Dom}(g)$
  - $\text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g)$
  - $\text{Dom}(f/g) = [\text{Dom}(f) \cap \text{Dom}(g)] \setminus \{g(x) = 0\}$ [i.e. Find $\text{Dom}(f) \cap \text{Dom}(g)$, then exclude x's where $g(x) = 0$]
- **LEVEL CURVES:** $(k \in \mathbb{R})$
  - Given explicit 3D surface $z = f(x, y)$, the **level curves** are the curves $f(x, y) = k$ projected onto the xy-plane.
  - Given implicit 3D surface $F(x, y, z) = 0$, the **level curves** are intersections of the surface with the planes $z = k$ projected onto the xy-plane.
- **LEVEL SURFACES:** $(k \in \mathbb{R})$
  - Given 4D hypersurface $w = f(x, y, z)$, the **level surfaces** are the surfaces $f(x, y, z) = k$ projected onto xyz-space.

EX 11.1.1: Let $f(x,y) = x^3 e^y$. Compute: (a) $f(0,0)$ (b) $f(2,0)$ (c) $f(0,2)$ (d) $f\left(t,t^2\right)$ (e) $\frac{d}{dt} f\left(t,t^2\right)$

EX 11.1.2: Let $f(x,y,z) = x + y^2 - z^3$. Compute: (a) $f(1,2,3)$ (b) $f(3,2,1)$ (c) $f\left(t,\sqrt{t},e^t\right)$, where $t \geq 0$

EX 11.1.3: Find or sketch the domain of $f(x, y) = x^2 - xy + y^2$.

EX 11.1.4: Find or sketch the domain of $f(x, y) = e^{x+y} - \dfrac{1}{x^2 - y^2}$.

EX 11.1.5: Find or sketch the domain of $f(x, y) = \sqrt{1 - x^2 - y^2}$.

EX 11.1.6: Find or sketch the domain of $f(x, y) = \dfrac{1}{(x - 3)(y + 5)} - \ln x + \sqrt{y}$.

EX 11.1.7: Let $f(x, y) = 3x - 5y$. Identify & characterize the level curve $f(x, y) = 7$.

EX 11.1.8: Let $g(x, y) = 5x^2 - y^2$. Identify & characterize the level curve $g(x, y) = 5$.

EX 11.1.9: Let $f(x, y, z) = x - y - z$. Identify & characterize the level surface $f(x, y, z) = 1$.

EX 11.1.10: Let $h(x, y, z) = 3x^2 + 2y^2 - z$. Identify & characterize the level surface $h(x, y, z) = 0$.