FUNCTIONS OF SEVERAL VARIABLES: LIMITS & CONTINUITY [SST 11.2]

- **THE FUNCTION LANDSCAPE (SO FAR):**

FUNCTION TYPE	PROTOTYPE	MAPPING	GRAPH	FIRST SEEN
(Scalar) Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$	2D Curve in xy-plane	Algebra
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$	3D Curve in xyz-space	Calc III (Ch10)
(Scalar) Function of 2 Variables	$z = f(x, y)$	$f : \mathbb{R}^2 \to \mathbb{R}$	3D Surface in xyz-space	Calc III (Ch11)
(Scalar) Function of 3 Variables	$w = f(x, y, z)$	$f : \mathbb{R}^3 \to \mathbb{R}$	4D Hypersurface in $xyzw$-space	Calc III (Ch11)

- **FUNCTIONS OF TWO VARIABLES (EVALUATING LIMITS):** $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$

1. Try **naive substitution (NS)**: Try to compute $f(x_0, y_0)$.
2. If naive substitution yields a finite real number L, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$
3. If naive substitution yields **indeterminate form** $\frac{0}{0}$, $\pm\frac{\infty}{\infty}$, $0 \cdot \infty$, or $\infty - \infty$, then consider the following:
   - If $(x_0, y_0) = (0, 0)$ AND $f(x, y)$ contains $\left(x^2 + y^2\right)$ expression(s), consider converting to **polar coordinates**:
     $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$
   - Factor polynomial(s) & cancel like factors
   - Rationalize the numerator or denominator
   - Combine fractions
   - Throw a factor "downstairs" e.g. $x^2 \ln x = \dfrac{\ln x}{1/x^2}$
   - Change of variables (CV): Let $u = g(x, y)$, then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ becomes $\lim\limits_{u\to u_0} h(u)$
     * In this form $\lim\limits_{u\to u_0} h(u)$, one can use any notion/method available from Calculus I such as **L'Hôpital's Rule** and **One-sided limits**.

- REMARK: **L'Hôpital's Rule** is useless for functions of several variables!
- REMARK: **One-sided limits** are meaningless for functions of several variables!

- **FUNCTIONS OF TWO VARIABLES (NON-EXISTENCE OF LIMITS):**
  - The limit $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$ holds regardless of which path was taken to approach (x_0, y_0)
  - To **show a limit does not exist**, show that two different paths to the point yield different limits.

- **FUNCTIONS OF TWO VARIABLES (CONTINUITY):**
  - Function $f(x, y)$ is **continuous at point** (x_0, y_0) if:

    $f(x_0, y_0)$ exists, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, AND $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$
  - Function $f(x, y)$ is **continuous on set** $S \iff f$ is continuous at each point in S.
  - Similar statements hold for functions of 3 variables.

EX 11.2.1: Evaluate $\lim_{(x,y)\to(-1,2)} \left(x^3 + x^2y^2 - xy + y^2 - 1\right)$

EX 11.2.2: Evaluate $\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y}$

EX 11.2.3: Evaluate $\lim_{(x,y)\to(c,c)} \frac{x^2 - y^2}{x - y}$, where $c \in \mathbb{R}$

EX 11.2.4: Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$

EX 11.2.5: Evaluate $\lim_{(x,y,z)\to(1,3,2)} 3^z \ln\left(1 - x + e^{xy}\right)$

EX 11.2.6: Evaluate $\lim_{(x,y,z)\to(1,-2,3)} \frac{\left(x^2-1\right)\left(y^2-4\right)\left(z^2-\pi^2\right)}{(x-1)(y+2)(z-\pi)}$

EX 11.2.7: Evaluate $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{3-\sqrt{x^2+y^2+z^2+9}}$

EX 11.2.8: Evaluate $\lim_{(x,y)\to(0,0)} \frac{\tan\left(x^2+y^2\right)}{x^2+y^2}$

EX 11.2.9: Evaluate $\lim_{(x,y)\to(0,0)} x \ln \sqrt{x^2+y^2}$

EX 11.2.10: Show that $\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$ does not exist.

EX 11.2.11: Show that $\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x+y^2}$ does not exist.

EX 11.2.12: Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$ does not exist.

EX 11.2.13: Let $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } (x,y) = (0,0) \end{cases}$. Is f continuous at $(0,0)$? (Justify answer)

EX 11.2.14: Let $g(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & \text{, for } (x,y) \neq (0,0) \\ 0 & \text{, for } (x,y) = (0,0) \end{cases}$. Is g continuous at $(0,0)$? (Justify answer)

EX 11.2.15: Let $h(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{, for } (x,y) \neq (0,0) \\ k & \text{, for } (x,y) = (0,0) \end{cases}$, where $k \in \mathbb{R}$.

What value of k ensures that h is continuous at $(0,0)$?