FUNCTIONS OF SEVERAL VARIABLES: PARTIAL DERIVATIVES [SST 11.3]

- **1^{st}-ORDER PARTIAL DERIVATIVES (DEFINITIONS):** Given function $f(x, y)$:
  - ⋆ $\dfrac{\partial f}{\partial x} := \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ $\quad$ $\dfrac{\partial f}{\partial y} := \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$
  - ⋆ $\dfrac{\partial f}{\partial x}$ measures the (instantaneous) rate of change of f as x changes, holding the other variable y fixed (constant).
  - ⋆ $\dfrac{\partial f}{\partial y}$ measures the (instantaneous) rate of change of f as y changes, holding the other variable x fixed (constant).

- **1^{st}-ORDER PARTIAL DERIVATIVES (RULES):**
  - Use any of the ordinary derivative rules from Calculus I, just **treat the other independent variables as constant**.

- **2^{nd}-ORDER PARTIAL DERIVATIVES:** Given function $f(x, y)$:

$$\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = (f_x)_x = f_{xx}$$
$$\frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = (f_x)_y = f_{xy}$$
$$\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = (f_y)_x = f_{yx}$$
$$\frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = (f_y)_y = f_{yy}$$

- **3^{rd}-ORDER PARTIAL DERIVATIVES:** Given function $f(x, y)$:

$$\frac{\partial^3 f}{\partial x^3} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x^2}\right] = (f_{xx})_x = f_{xxx}$$
$$\frac{\partial^3 f}{\partial y \partial x^2} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial x^2}\right] = (f_{xx})_y = f_{xxy}$$
$$\frac{\partial^3 f}{\partial x \partial y \partial x} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = (f_{xy})_x = f_{xyx}$$
$$\frac{\partial^3 f}{\partial y^2 \partial x} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = (f_{xy})_y = f_{xyy}$$
$$\frac{\partial^3 f}{\partial y^3} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial y^2}\right] = (f_{yy})_y = f_{yyy}$$
$$\frac{\partial^3 f}{\partial x \partial y^2} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y^2}\right] = (f_{yy})_x = f_{yyx}$$
$$\frac{\partial^3 f}{\partial y \partial x \partial y} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial x \partial y}\right] = (f_{yx})_y = f_{yxy}$$
$$\frac{\partial^3 f}{\partial x^2 \partial y} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x \partial y}\right] = (f_{yx})_x = f_{yxx}$$

- **TOTAL DIFFERENTIALS:**
  - Given $f(x, y)$, then **total differential** $df := \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$
  - Given $f(x, y, z)$, then **total differential** $df := \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz$

- **ERROR APPROXIMATION:**
  - Given $f(x, y)$ and "small" errors $\Delta x, \Delta y$ in x, y. Then **linear error** $\Delta f \approx \dfrac{\partial f}{\partial x}\Delta x + \dfrac{\partial f}{\partial y}\Delta y$
  - Given $f(x, y, z)$ and "small" errors $\Delta x, \Delta y, \Delta z$ in x, y, z. Then **linear error** $\Delta f \approx \dfrac{\partial f}{\partial x}\Delta x + \dfrac{\partial f}{\partial y}\Delta y + \dfrac{\partial f}{\partial z}\Delta z$

- **CONTINUITY OF PARTIAL DERIVATIVES:**
  - Given $f(x, y)$, then $f \in C^{(1,1)}(S) \iff f, f_x, f_y \in C(S)$
  - Given $f(x, y)$, then $f \in C^{(2,2)}(S) \iff f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx} \in C(S)$
  - Given $f(x, y)$, then f is **differentiable** on set $S \subseteq \mathbb{R}^2$ if $f \in C^{(1,1)}(S)$.
  - Given $f(x, y, z)$, then $f \in C^{(1,1,1)}(S) \iff f, f_x, f_y, f_z \in C(S)$
  - Given $f(x, y, z)$, then $f \in C^{(2,2,2)}(S) \iff f, f_x, f_y, f_z, f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy} \in C(S)$
  - Given $f(x, y, z)$, then f is **differentiable** on set $S \subseteq \mathbb{R}^3$ if $f \in C^{(1,1,1)}(S)$.

FUNCTIONS OF SEVERAL VARIABLES: PARTIAL DERIVATIVES [SBS 11.3]

- **1^{st}-ORDER PARTIAL DERIVATIVES (DEFINITIONS):** Given function $f(x, y, z)$:
  - $\star$ $\dfrac{\partial f}{\partial x} := \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$ $\quad \dfrac{\partial f}{\partial y} := \lim\limits_{\Delta y \to 0} \dfrac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y}$ $\quad \dfrac{\partial f}{\partial z} := \lim\limits_{\Delta z \to 0} \dfrac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z}$
  - $\star$ $\dfrac{\partial f}{\partial x}$ measures the (instantaneous) rate of change of f as x changes, holding the other variables y, z fixed (constant).
  - $\star$ $\dfrac{\partial f}{\partial y}$ measures the (instantaneous) rate of change of f as y changes, holding the other variables x, z fixed (constant).
  - $\star$ $\dfrac{\partial f}{\partial z}$ measures the (instantaneous) rate of change of f as z changes, holding the other variables x, y fixed (constant).

- **1^{st}-ORDER PARTIAL DERIVATIVES (RULES):**
  - Use any of the ordinary derivative rules from Calculus I, just **treat the other independent variables as constant**.

- **2^{nd}-ORDER PARTIAL DERIVATIVES:** Given function $f(x, y, z)$:

$$\frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = (f_x)_x = f_{xx}$$
$$\frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = (f_y)_y = f_{yy}$$
$$\frac{\partial^2 f}{\partial z^2} := \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z}\right] = (f_z)_z = f_{zz}$$
$$\frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = (f_x)_y = f_{xy}$$
$$\frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = (f_y)_x = f_{yx}$$
$$\frac{\partial^2 f}{\partial z \partial x} := \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial x}\right] = (f_x)_z = f_{xz}$$
$$\frac{\partial^2 f}{\partial x \partial z} := \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial z}\right] = (f_z)_x = f_{zx}$$
$$\frac{\partial^2 f}{\partial z \partial y} := \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial y}\right] = (f_y)_z = f_{yz}$$
$$\frac{\partial^2 f}{\partial y \partial z} := \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right] = (f_z)_y = f_{zy}$$

- **3^{rd}-ORDER PARTIAL DERIVATIVES:** Given function $f(x, y, z)$:
  - There are 27 3^{rd}-order partials – too many to list! Here are seven of them:

$$\frac{\partial^3 f}{\partial x^3} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xxx}$$
$$\frac{\partial^3 f}{\partial z^3} := \frac{\partial}{\partial z}\left[\frac{\partial^2 f}{\partial z^2}\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z}\right]\right] = f_{zzz}$$
$$\frac{\partial^3 f}{\partial z \partial y^2} := \frac{\partial}{\partial z}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\right] = f_{yyz}$$
$$\frac{\partial^3 f}{\partial x^2 \partial z} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x \partial z}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial z}\right]\right] = f_{zxx}$$
$$\frac{\partial^3 f}{\partial x \partial z \partial x} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial z \partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xzx}$$
$$\frac{\partial^3 f}{\partial x \partial y \partial z} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y \partial z}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right]\right] = f_{zyx}$$
$$\frac{\partial^3 f}{\partial z \partial y \partial x} := \frac{\partial}{\partial z}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xyz}$$

EX 11.3.1: Let $z = x^4y^5 - xy - 2x - 3y + 100.$ Find:

(a) $\dfrac{\partial z}{\partial x} =$

(b) $\dfrac{\partial z}{\partial y} =$

(c) $\dfrac{\partial^2 z}{\partial x^2} =$

(d) $\dfrac{\partial^2 z}{\partial y^2} =$

(e) $\dfrac{\partial^2 z}{\partial x \partial y} =$

(f) $\dfrac{\partial^3 z}{\partial x^3} =$

(g) $\dfrac{\partial^3 z}{\partial x^2 \partial y} =$

EX 11.3.2: Let $f(x,y,z) = \cos(\pi x/4)\sin(\pi y/3)\arctan\left(z^2\right)$. Find:

(a) $f_x(3,1,-1)$

(b) $\left.\dfrac{\partial f}{\partial y}\right|_{(3,1,-1)}$

(c) $\left.\dfrac{\partial f}{\partial z}\right|_{(x,y,z)=(3,1,-1)}$

EX 11.3.3: Let $g(x,y) = \dfrac{e^{xy}}{1+xy}$. Find:

(a) $g_x =$

(b) $\dfrac{\partial g}{\partial y} =$

EX 11.3.4: Let $h(x, y, z) = xz \ln(x + y)$, where $x, y > 0$. Find:

(a) $\dfrac{\partial h}{\partial x} =$

(b) $\dfrac{\partial h}{\partial y} =$

(c) $h_z =$

EX 11.3.5: Let $T(x, y, t)$ model temperature of a plate, where $x \equiv$ x-coord. on plate, $y \equiv$ y-coord on plate, and $t \equiv$ time.

(a) Interpret what $\dfrac{\partial T}{\partial x}$ measures in this context.

(b) Interpret what $\dfrac{\partial T}{\partial y}$ measures in this context.

(c) Interpret what $\dfrac{\partial T}{\partial t}$ measures in this context.

EX 11.3.6: Recall the formula for the volume V of a right-circular cone with radius r and height h: $V = \dfrac{1}{3}\pi r^2 h$.

(a) Compute the rate of change in volume as the radius changes, holding the height fixed (constant).

(b) Compute the rate of change in volume as the height changes, holding the radius fixed (constant).

EX 11.3.7: Let $f(x,y,z) = x^3y^2z - xyz + 3xz$. Show that f is differentiable on $\mathbb{R}^3$.

EX 11.3.8: Let $g(x,y) = 2xy$.

(a) Compute the total differential of g.

(b) Let Δx, Δy be "small" errors of x, y respectively. Find the linear error Δg.

EX 11.3.9: Let $h(x,y,z) = xy^2z^3$.

(a) Compute the total differential of h.

(b) Let Δx, Δy, Δz be "small" errors in x, y, z respectively. Find the linear error Δh.

EX 11.3.10: Let $f(x,y,z) = x^2y^3z^4$. Compute:

(a) $\dfrac{\partial^2 f}{\partial x^2} =$

(b) $\dfrac{\partial^2 f}{\partial z \partial x} =$

(c) $\dfrac{\partial^3 f}{\partial z^3} =$

(d) $\dfrac{\partial^3 f}{\partial y^2 \partial x} =$

(e) $\dfrac{\partial^3 f}{\partial x \partial z^2} =$

(f) $\dfrac{\partial^3 f}{\partial x \partial y \partial z} =$