# FUNCTIONS OF SEVERAL VARIABLES: CHAIN RULES [SST 11.5]

- **THE "1-1" CHAIN RULE (FROM CALCULUS I):**
  - Let $y = f(x) \in C^1$ where $x = g(t) \in C^1$. Then: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

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- **THE "1-2" CHAIN RULE:**
  - Let $z = f(x) \in C^1$ where $x = g(s,t) \in C^{(1,1)}$. Then: $\dfrac{\partial z}{\partial s} = \dfrac{dz}{dx}\dfrac{\partial x}{\partial s}$ $\qquad \dfrac{\partial z}{\partial t} = \dfrac{dz}{dx}\dfrac{\partial x}{\partial t}$
- **THE "2-1" CHAIN RULE:**
  - Let $z = f(x,y) \in C^{(1,1)}$ where $x = g(t) \in C^1$ and $y = h(t) \in C^1$. Then: $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$
- **THE "2-2" CHAIN RULE:**
  - Let $z = f(x,y) \in C^{(1,1)}$ where $x = g(s,t) \in C^{(1,1)}$ and $y = h(s,t) \in C^{(1,1)}$. Then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

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- **THE "1-3" CHAIN RULE:**
  - Let $z = f(x) \in C^1$ where $x = g(r,s,t) \in C^{(1,1,1)}$. Then: $\dfrac{\partial z}{\partial r} = \dfrac{dz}{dx}\dfrac{\partial x}{\partial r}$ $\qquad \dfrac{\partial z}{\partial s} = \dfrac{dz}{dx}\dfrac{\partial x}{\partial s}$ $\qquad \dfrac{\partial z}{\partial t} = \dfrac{dz}{dx}\dfrac{\partial x}{\partial t}$
- **THE "3-1" CHAIN RULE:**
  - Let $w = f(x,y,z) \in C^{(1,1,1)}$ s.t. $x = g(t) \in C^1$, $y = h(t) \in C^1$, $z = p(t) \in C^1$. Then: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$
- **THE "2-3" CHAIN RULE:**
  - Let $z = f(x,y) \in C^{(1,1)}$ s.t. $x = g(r,s,t) \in C^{(1,1,1)}$, $y = h(r,s,t) \in C^{(1,1,1)}$. Then:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- **THE "3-2" CHAIN RULE:**
  - Let $w = f(x,y,z) \in C^{(1,1,1)}$ s.t. $x = g(s,t)$, $y = h(s,t)$, $z = p(s,t) \in C^{(1,1)}$. Then:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

- **THE "3-3" CHAIN RULE:**
  - Let $w = f(x,y,z) \in C^{(1,1,1)}$ s.t. $x = g(r,s,t)$, $y = h(r,s,t)$, $z = p(r,s,t) \in C^{(1,1,1)}$. Then:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

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- **IMPLICIT DIFFERENTIATION (2-VARIABLE FUNCTION):**
  - Let $F(x,y) = 0$ s.t. $F \in C^{(1,1)}$ and $y$ is implicitly a function of $x$. Then:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}, \text{ provided } F_y \neq 0$$

- **IMPLICIT DIFFERENTIATION (3-VARIABLE FUNCTION):**
  - Let $F(x,y,z) = 0$ s.t. $F \in C^{(1,1,1)}$ and $z$ is implicitly a function of $(x,y)$. Then:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \text{ provided } F_z \neq 0$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \text{ provided } F_z \neq 0$$

**EX 11.5.1:** Let $z = x^2y - xy^2$, $x = 3 + t^5$, $y = 5 - t^2$.

(a) Sketch the dependency tree for $z$.

(b) Write out the Multivariable Chain Rule for $\frac{dz}{dt}$.

$\frac{dz}{dt} =$

(c) Using the Chain Rule found in part (b), compute $\frac{dz}{dt}$.

(d) Compute $\frac{dz}{dt}$ when $t = 1$.

**EX 11.5.2:** Let $w = xe^{y/z}$, $x = 2 + t^2$, $y = 1 - t$, $z = 3 - 2t$.

(a) Sketch the dependency tree for $w$.

(b) Write out the Multivariable Chain Rule for $\frac{dw}{dt}$.

$\frac{dw}{dt} =$

(c) Using the Chain Rule found in part (b), compute $\frac{dw}{dt}$.

(d) Compute $\frac{dw}{dt}$ when $t = 1$.

**EX 11.5.3:** Let $z = e^{xy}\tan(\pi x)$, $x = s - 3t$, $y = st$.

(a) Sketch the dependency tree for $z$.

(b) Write out the Multivariable Chain Rule for $\frac{\partial z}{\partial t}$.

$\frac{\partial z}{\partial t} =$

(c) Using the Chain Rule found in part (b), compute $\frac{\partial z}{\partial t}$.

(d) Compute $\frac{\partial z}{\partial t}$ when $s = 1$ and $t = 2$.

**EX 11.5.4:** Let $u = z^2 - xy$, $x = pr\sin\theta$, $y = r\cos\theta$, $z = p - r$.

(a) Sketch the dependency tree for $u$.

(b) Write out the Multivariable Chain Rule for $\dfrac{\partial u}{\partial p}$.

$\dfrac{\partial u}{\partial p} =$

(c) Using the Chain Rule found in part (b), compute $\dfrac{\partial u}{\partial p}$.

(d) Compute $\dfrac{\partial u}{\partial p}$ when $p = 2$, $r = 3$, and $\theta = \pi$.

**EX 11.5.5:** Let $v = \sqrt{r^2 + s^2}$, $r = 4y + x\cos(3t)$, $s = x - y\sin(2t)$.

(a) Sketch the dependency tree for $v$.

(b) Write out the Multivariable Chain Rule for $\frac{\partial v}{\partial x}$.

$\frac{\partial v}{\partial x} =$

(c) Using the Chain Rule found in part (b), compute $\frac{\partial v}{\partial x}$.

(d) Compute $\frac{\partial v}{\partial x}$ when $x = 1$, $y = 2$, and $t = 0$.

**11.5.6:** Let $z = \arctan x$, $x = \dfrac{r - st}{\sqrt{3}}$.

(a) Sketch the dependency tree for $z$.

(b) Write out the Multivariable Chain Rule for $\dfrac{\partial z}{\partial s}$.

$\dfrac{\partial z}{\partial s} =$

(c) Using the Chain Rule found in part (b), compute $\dfrac{\partial z}{\partial s}$.

(d) Compute $\dfrac{\partial z}{\partial s}$ when $r = 3$, $s = 2$, and $t = 2$.

**EX 11.5.7:** Given implicit function $\sin(x+5y) = 7xe^{-y}$, use a Multivariable Chain Rule to find $\dfrac{dy}{dx}$.

**EX 11.5.8:** Given implicit function $yz = \ln(x+z)$, use a Multivariable Chain Rule to find $\dfrac{\partial z}{\partial x}$ & $\dfrac{\partial z}{\partial y}$.

**EX 11.5.9:** A rectangular box is changing in such a way that its length $\ell$ is increasing at a rate of 3 ft/hr, its width $w$ is increasing at a rate of 5 ft/hr, and its height $h$ is decreasing at a rate of 2 ft/hr.

At what rate is the volume of the box changing when its length is 10 ft, its width is 20 ft, and its height is 15 ft?