FUNCTIONS OF SEVERAL VARIABLES: GRADIENTS [SST 11.6]

- **GRADIENTS:**
  - Let $f(x, y) \in C^{(1,1)}$. Then the **gradient** of f is a vector in $\mathbb{R}^2$ given by:

$$\text{grad } f \equiv \nabla f(x, y) \equiv \nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = f_x \hat{\mathbf{i}} + f_y \hat{\mathbf{j}}$$

  - Let $f(x, y, z) \in C^{(1,1,1)}$. Then the **gradient** of f is a vector in $\mathbb{R}^3$ given by:

$$\text{grad } f \equiv \nabla f(x, y, z) \equiv \nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = f_x \hat{\mathbf{i}} + f_y \hat{\mathbf{j}} + f_z \hat{\mathbf{k}}$$

- **DIRECTIONAL DERIVATIVES:**
  - Let $f(x, y) \in C^{(1,1)}$. Then, the **directional derivative** of f at point $P_0(x_0, y_0)$ in the direction of unit vector $\hat{\mathbf{v}} \in \mathbb{R}^2$ is:

$$D_{\mathbf{v}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{\mathbf{v}}$$

  - Let $f(x, y, z) \in C^{(1,1,1)}$. Then, the **directional derivative** of f at point $P_0(x_0, y_0, z_0)$ in the direction of unit vector $\hat{\mathbf{v}} \in \mathbb{R}^3$ is:

$$D_{\mathbf{v}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \hat{\mathbf{v}}$$

- **GRADIENT (STEEPEST ASCENT/DESCENT)**
  - Given surface $z = f(x, y)$ such that $f \in C^{(1,1)}$. Then:

    (i) ∇f is **normal** to a level curve that contains point (x, y).

    (ii) ∇f points in the (compass) direction of **steepest ascent** of f from (x, y).

    (iii) $-\nabla f$ points in the direction of **steepest descent** of f from point (x, y).

    (iv) $||\nabla f||$ is the **maximum rate of change** of f at point (x, y).

- **TANGENT PLANE & NORMAL LINE TO A LEVEL SURFACE $F(x, y, z) = k$:**
  - Given **level surface** $F(x, y, z) = k$, where $k \in \mathbb{R}$ and $F \in C^{(1,1,1)}$.

    Let $P_0(x_0, y_0, z_0)$ be a point on the level surface s.t. $\nabla F(x_0, y_0, z_0) \neq \vec{\mathbf{0}}$. Then:

    The equation of the **tangent plane** $\mathbb{T}$ to the level surface at point P_0 is:

$$\mathbb{T} : F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

    The equation of the **normal line** ℓ to the level surface at point P_0 is:

$$\ell(t) = \langle x_0 + F_x(x_0, y_0, z_0)t, y_0 + F_y(x_0, y_0, z_0)t, z_0 + F_z(x_0, y_0, z_0)t \rangle$$

- **TANGENT PLANE & NORMAL LINE TO A SURFACE $z = f(x, y)$:**
  - First, write surface as a level surface: $z = f(x, y) \iff z - f(x, y) = 0 \iff F(x, y, z) := z - f(x, y) = 0$
  - Then follow the procedure above.

EX 11.6.1: Let $f(x, y) = \sin(2x + 3y)$.

(a) Compute the gradient ∇f.

(b) Compute $\nabla f(-6, 4)$.

(c) Compute the directional derivative $D_{\mathbf{v}} f(-6, 4)$, where $\mathbf{v} = \langle \sqrt{3}, -1 \rangle$.

EX 11.6.2: Let $g(x, y, z) = xe^y + ye^z + ze^x$.

(a) Compute the gradient ∇g.

(b) Compute $\nabla g(0, 0, 0)$.

(c) Compute the directional derivative $D_{\mathbf{v}} g(0, 0, 0)$, where $\mathbf{v} = \langle 5, 1, -2 \rangle$.

EX 11.6.3: Let $f(x, y) = \ln \sqrt{x^2 + y^2}$.

(a) In what direction is f increasing most rapidly from point $P(1, 2)$?

(b) What is the maximum rate of increase of f from point $P(1, 2)$?

(c) In what direction is f decreasing most rapidly from point $P(1, 2)$?

EX 11.6.4: Let $h(x, y, z) = \tan(x + 2y + 3z)$.

(a) In what direction is h increasing most rapidly from point $P(-5, 1, 1)$?

(b) What is the maximum rate of increase of h from point $P(-5, 1, 1)$?

(c) In what direction is h decreasing most rapidly from point $P(-5, 1, 1)$?

EX 11.6.5: Find a unit vector that's normal to the hyperbola $\frac{x^2}{5} - \frac{y^2}{3} = 1$ at point $Q(\sqrt{10}, \sqrt{3})$.

EX 11.6.6: Given level surface $x^4 + y^4 + z^4 = 3$:

(a) Find an equation of the tangent plane to the level surface at point $P(1, -1, -1)$.

(b) Find an equation of the normal line to the level surface at point $P(1, -1, -1)$.

(c) Find a unit normal vector to the level surface at point $P(1, -1, -1)$.

EX 11.6.7: Given surface $z = \sin(xy)$:

(a) Find an equation of the tangent plane to the surface at point $P(\sqrt{\pi}, \sqrt{\pi}, 0)$.

(b) Find an equation of the normal line to the surface at point $P(\sqrt{\pi}, \sqrt{\pi}, 0)$.

(c) Find a unit normal vector to the surface at point $P(\sqrt{\pi}, \sqrt{\pi}, 0)$.