FUNCTIONS OF TWO VARIABLES: RELATIVE EXTREMA [SST 11.7]

- **OPEN, CLOSED, AND BOUNDED SETS IN $\mathbb{R}^2$:**
  - A set $S \subseteq \mathbb{R}^2$ is **open** if S contains none of its boundary.
  - A set $S \subseteq \mathbb{R}^2$ is **closed** if S contains all of its boundary.
  - $\mathbb{R}^2$ and $\emptyset$ (empty set) are **both open and closed**.
  - **Open disk** centered at (x_0, y_0) with radius $r > 0$: $\mathbb{D}(x_0, y_0; r) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$
  - **Closed disk** centered at (x_0, y_0) with radius $r > 0$: $\overline{\mathbb{D}}(x_0, y_0; r) := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$
  - A set $S \subset \mathbb{R}^2$ is **bounded** if S is contained in an open disk.

- **SUFFICIENT CONDITION FOR EQUALITY OF MIXED 2^{nd}-ORDER PARTIALS:**
  - Let $f(x, y) \in C^{(2,2)}$. Then $f_{xy} = f_{yx}$

- **CRITICAL POINTS:**
  - Let $f(x, y)$ be defined on an open set $S \subseteq \mathbb{R}^2$ such that $(x_0, y_0) \in S$.
    Then (x_0, y_0) is a **critical point** of f is either one of the following is true:
    (i) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
    (ii) At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ DNE

- **RELATIVE MIN's, RELATIVE MAX's, SADDLE POINTS ("FIRST PRINCIPLES" DEFINITIONS):**
  - Let $f(x, y)$ be defined on an open set $S \subseteq \mathbb{R}^2$ such that $(x_0, y_0) \in S$. Then:

    (x_0, y_0) is a **relative maximum** if $f(x, y) \leq f(x_0, y_0) \;\; \forall (x, y) \in \mathbb{D}(x_0, y_0; r)$.
    (x_0, y_0) is a **relative minimum** if $f(x, y) \geq f(x_0, y_0) \;\; \forall (x, y) \in \mathbb{D}(x_0, y_0; r)$.
    (x_0, y_0) is a **saddle point** if $\exists (x_1, y_1), (x_2, y_2) \in \mathbb{D}(x_0, y_0; r)$ s.t. $f(x_1, y_1) > f(x_0, y_0)$ and $f(x_2, y_2) < f(x_0, y_0)$.

- **RELATIVE MIN's, RELATIVE MAX's, SADDLE POINTS (2^{nd}-ORDER PARTIALS TEST):**
  - Let $f(x, y) \in C^{(2,2)}\Big(\mathbb{D}(x_0, y_0; r)\Big)$ s.t. f has a critical point at (x_0, y_0).
    Form the **discriminant** of f: $\Delta(x, y) := \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = f_{xx} f_{yy} - (f_{xy})^2$
    Then:
    (x_0, y_0) is a **relative max** if $\Big(\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0\Big)$ OR $\Big(\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0\Big)$

    (x_0, y_0) is a **relative min** if $\Big(\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0\Big)$ OR $\Big(\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0\Big)$

    (x_0, y_0) is a **saddle point** if $\Delta(x_0, y_0) < 0$

    The test is **inconclusive** if $\Delta(x_0, y_0) = 0$.
    So, apply the **"First Principles" definitions** of rel min, rel max, and saddle point. (see above)

FUNCTIONS OF TWO VARIABLES: ABSOLUTE EXTREMA [SST 11.7]

- **ABSOLUTE EXTREMA (DEFINITIONS):**

  - Given function $f(x,y)$:

    (x_M, y_M) is an **absolute maximum** of f if $f(x_M, y_M) \geq f(x,y) \quad \forall (x,y) \in \text{ Dom}(f)$

    (x_m, y_m) is an **absolute minimum** of f if $f(x_m, y_m) \leq f(x,y) \quad \forall (x,y) \in \text{ Dom}(f)$

    If (x_M, y_M) is an abs max of f, then $f(x_M, y_M)$ is the **absolute max value** of f.

    If (x_m, y_m) is an abs min of f, then $f(x_m, y_m)$ is the **absolute min value** of f.

    The **extreme values** of f are the **abs max value & abs min value** of f.

- **EXTREME VALUE THEOREM (E-V-T):**

  - Let $f(x,y) \in C(S)$ where set $S \subset \mathbb{R}^2$ is **closed & bounded**.

    Then f attains extreme values over the set S.

- **ABSOLUTE EXTREMA (PROCEDURE):**

  - Let $f(x,y) \in C(S)$ where set $S \subset \mathbb{R}^2$ is **closed & bounded**.

    Then to find the absolute extrema of f over S, follow this procedure:

    * Find all critical points (CP's) of f.

    * Sketch & identify all **boundary curves (BC's) & boundary points (BP's)** of S.

    A **boundary point** is the **intersection** of two boundary curves.

    * Discard any critical points (CP's) that are <u>not</u> in S.

    * Find all points on the **boundary** of S where absolute extrema can occur (**boundary critical points (BCP's)**).

    To do this, find the absolute extrema on a function of <u>one variable</u> by plugging in one of the BC's of S.

    Sometimes it's best to **parameterize** the BC.

    Repeat for each BC.

    * Build a **table** by computing f for each CP, BP, and BCP.

    * The abs max value of f is the largest of all computed values in table.

    * The abs min value of f is the smallest of all computed values in table.

EX 11.7.1: Let $f(x, y) = 3x - x^3 - 3xy^2$. Find & classify all critical points of f.

EX 11.7.2: Let $g(x, y) = x \sin y$. Find & classify all critical points of g.

EX 11.7.3: Let $h(x, y) = x^6 + y^8$. Find & classify all critical points of h.

EX 11.7.4: Let $f(x,y) = 3x - x^3 - 3xy^2$ and set $S = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2\}$.

Find the extreme values of f over S and the points at which they occur.

EX 11.7.5: Let $g(x, y) = x^2 + xy + y^2$ and set $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Find the extreme values of g over S and the points at which they occur.

EX 11.7.6: Let $h(x, y) = 2 + 2x + 2y - x^2 - y^2$ and set S be the closed triangle with vertices $(0, 0)$, $(0, 8)$, $(8, 0)$. Find the extreme values of h over S and the points at which they occur.