CONSTRAINED OPTIMIZATION: LAGRANGE MULTIPLIERS [SST 11.8]

- **LAGRANGE MULTIPLIERS (FUNCTIONS OF TWO VARIABLES):**

  - Let $f, g \in C^{(1,1)}$ s.t. $f(x, y)$ has an extremum subject to constraint $g(x, y) = k$, where $k \in \mathbb{R}$.

    Then to find the extreme value(s) of f:

    Solve for x, y, λ: $\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y) = k \end{cases}$

    or equivalently,

    STEP 1: Solve for x, y, λ: $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}$

    The resulting solutions (x, y) are called the **constrained critical points (CCP's)** of f.

    STEP 2: Build a table computing f for each constrained critical point (CCP)

    If there's only one CCP, pick any other (simple) point on the constraint to compare with.

    STEP 3: Compare values in the table to determine the extreme value(s) of f.

- **LAGRANGE MULTIPLIERS (FUNCTIONS OF THREE VARIABLES):**

  - Let $f, g \in C^{(1,1,1)}$ s.t. $f(x, y, z)$ has an extremum subject to constraint $g(x, y, z) = k$, where $k \in \mathbb{R}$.

    Then to find the extreme value(s) of f:

    Solve for x, y, z, λ: $\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = k \end{cases}$

    or equivalently,

    STEP 1: Solve for x, y, z, λ: $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x, y, z) = k \end{cases}$

    The resulting solutions (x, y, z) are called the **constrained critical points (CCP's)** of f.

    STEP 2: Build a table computing f for each constrained critical point (CCP)

    If there's only one CCP, pick any other (simple) point on the constraint to compare with.

    STEP 3: Compare values in the table to determine the extreme value(s) of f.

EX 11.8.1: Using one Lagrange Multiplier λ, minimize $f(x,y) = 3x + y + 10$ subject to constraint $x^2y = 12$, where $x \geq 0$.

EX 11.8.2: Using one Lagrange Multiplier λ, maximize $f(x, y) = x^2 - y^2$ subject to constraint $x^2 + y^2 = 9$.

EX 11.8.3: Using one Lagrange Multiplier λ, minimize $f(x, y, z) = x - 2y + z$ subject to constraint $x^2 + 2y^2 + 3z^2 = 6$.

EX 11.8.4: Using one Lagrange Multiplier λ, maximize $f(x, y, z) = xyz$ subject to constraint $x + y + z = 1$, where $\begin{array}{c} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array}$.

EX 11.8.5: Find the largest product of positive real numbers x, y such that their sum is 60.

EX 11.8.6: Find the smallest sum of positive real numbers x, y, z such that their product is 64.