DOUBLE INTEGRALS: POLAR COORDINATES [SST 12.3]

- **SPECIAL POLAR CURVES:**

$\left(a \neq 0, b \neq 0, k \neq 0, n \in \mathbb{Z}_+\right)$ $\mathbb{Z}_+ \equiv$ The set of all **positive integers**.

POLAR CURVE	PROTOTYPE	REMARK(S)
Lines (Rays) thru Pole	$\theta = k$	Always Graph!
Horizontal Lines (Off-Pole)	$r = a\csc\theta$	Always convert!
Vertical Lines (Off-Pole)	$r = a\sec\theta$	Always convert!
Circles Centered at Pole	$r = k$	Always Graph!
Circles Containing Pole	$r = a\cos\theta, r = a\sin\theta$	Always Graph!
Cardioids	$r = a \pm a\cos\theta, r = a \pm a\sin\theta$	Always Graph!
Limaçons	$r = b \pm a\cos\theta, r = b \pm a\sin\theta$	Always Graph!
Roses	$r = a\cos(n\theta), r = a\sin(n\theta)$	Always Graph!

- **GRAPHING POLAR CURVES (PROCEDURE):**

1. Graph $r = f(\theta)$ on the usual xy-plane where $x = \theta$ & $y = r$ (**Rectangular Plot**)

   Use special angles for θ: $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$

   If $f(\theta)$ has a trig fcn, set its argument to these angles & solve for θ

2. Use the rectangular plot of $r = f(\theta)$ to trace the polar graph of $r = f(\theta)$ (**Polar Plot**)

IMPORTANT: Except for equations of lines, "connect the dots" using **smooth curves**, not line segments!

- **INTERSECTION OF TWO POLAR CURVES $r = f(\theta)$ & $r = g(\theta)$:**

  - Solving $f(\theta) = g(\theta)$ for θ finds some, but not necessarily all, intersection points.
  - In particular, intersections at the **pole** (origin) are nearly impossible to find algebraically because the pole has no single representation in polar coordinates that satisfies both $r = f(\theta)$ & $r = g(\theta)$.
  - Therefore, to find all intersection points, GRAPH BOTH CURVES!

- **AREA OF A RADIALLY SIMPLE (r-SIMPLE) POLAR REGION:**

  - Let D be a r-simple region s.t. $D = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta \text{ s.t. } 0 \leq \beta - \alpha \leq 2\pi\}$. Then:

$$\text{Area}(D) := \iint_D dA = \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Inner BC of } D}^{\text{Outer BC of } D} r\ dr\ d\theta = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} r\ dr\ d\theta$$

- **AREA OF A QUASI-RADIALLY SIMPLE (QUASI-r-SIMPLE) POLAR REGION:**

  - Let D be a r-simple region s.t. $D = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta \text{ for } g_1(\theta), \gamma \leq \theta \leq \delta \text{ for } g_2(\theta)\}$. Moreover, let BP's $(g_1(\alpha), \alpha), (g_2(\gamma), \gamma)$ share one ray & BP's $(g_1(\beta), \beta), (g_2(\delta), \delta)$ share another ray. Then:

$$\text{Area}(D) = \int_{\text{Smallest } \theta\text{-value for Outer BC}}^{\text{Largest } \theta\text{-value for Outer BC}} \int_{\text{Pole}}^{\text{Outer BC}} r\ dr\ d\theta - \int_{\text{Smallest } \theta\text{-value for Inner BC}}^{\text{Largest } \theta\text{-value for Inner BC}} \int_{\text{Pole}}^{\text{Inner BC}} r\ dr\ d\theta$$

$$\implies \text{Area}(D) = \int_\gamma^\delta \int_0^{g_2(\theta)} r\ dr\ d\theta - \int_\alpha^\beta \int_0^{g_1(\theta)} r\ dr\ d\theta$$

- **HOW TO SUBDIVIDE A POLAR REGION THAT'S NEITHER r-SIMPLE NOR QUASI-r-SIMPLE:**

  - Construct a ray that contains at least one BP that separates two outer BC's (or two inner BC's).
  - Repeat this as necessary.

- **CONVERTING A DOUBLE INTEGRAL FROM RECTANGULAR TO POLAR COORDINATES:**

  - Let $f(x,y) \in C(D)$ where D is a region that is either r-simple or quasi-r-simple (or can be subdivided). Then:

$$\iint_D f(x,y)\ dA = \iint_D f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$$

EX 12.3.1: Using polar coordinates, compute $I = \iint_D y^2 \, dA$, where region $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 3\}$.

EX 12.3.2: Using polar coordinates, compute $I = \iint_D \left(x^2 + y^2\right)\, dA$, where D is the disk $x^2 + y^2 \leq 4$ in Quadrant I.

EX 12.3.3:

(a) Graph the polar curve $r = 2 + 2\cos\theta$.

(b) Using double integral(s), find the area of the region enclosed by the polar curve $r = 2 + 2\cos\theta$.

EX 12.3.4:

(a) Graph the polar curve $r = 3\sin(2\theta)$.

(b) Using double integral(s), find the area of the region enclosed by the polar curve $r = 3\sin(2\theta)$.

EX 12.3.5:

(a) Graph the polar curves $r = 1$ and $r = 1 + \cos\theta$.

(b) Using double integral(s), find the area of the region inside both polar curves $r = 1$ and $r = 1 + \cos\theta$.

EX 12.3.6: Using polar coordinates, compute $I = \int_0^2 \int_0^{\sqrt{4-x^2}} x \; dy \; dx$.

EX 12.3.7: Using polar coordinates, compute $I = \int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} e^{x^2+y^2}\, dx\, dy$.

EX 12.3.8: Using double integral(s), find the volume of the solid E bounded by the paraboloid $z = 4 - x^2 - y^2$ and xy-plane.