DOUBLE INTEGRALS: SURFACE AREA [SST 12.4]

- **SURFACE AREA OF PORTION OF SURFACE $z = f(x, y)$:**
  - Let $f(x, y) \in C^{(1,1)}(D)$ in a region $D \subset \mathbb{R}^2$ of the xy-plane.
    Let S be the portion of the surface $z = f(x, y)$ that lies directly over D.
    (Region D is also known as the **projection** of S onto the xy-plane.)

    Then:

    $$\text{Surface Area}(S) := \iint_S \, dS = \iint_D \sqrt{1 + [f_x]^2 + [f_y]^2} \, dA = \iint_D \sqrt{1 + \left[\frac{\partial f}{\partial x}\right]^2 + \left[\frac{\partial f}{\partial y}\right]^2} \, dA$$
  - As usual, **sketch the region** D to help determine the limits of integration.
  - REMARK: The resulting double integral may be hard to compute.
    * If so, **rewrite double integral in polar coordinates**.

- **SURFACE AREA OF PORTION OF SURFACE $y = g(x, z)$ [OPTIONAL]:**
  - Let $g(x, z) \in C^{(1,1)}(D^{[xz]})$ in a region $D^{[xz]} \subset \mathbb{R}^2$ of the xz-plane.
    Let S be the portion of the surface $y = g(x, z)$ that lies directly over $D^{[xz]}$.
    (Region $D^{[xz]}$ is also known as the **projection** of S onto the xz-plane.)

    Then:

    $$\text{Surface Area}(S) := \iint_S \, dS = \iint_{D^{[xz]}} \sqrt{1 + [g_x]^2 + [g_z]^2} \, dA = \iint_{D^{[xz]}} \sqrt{1 + \left[\frac{\partial g}{\partial x}\right]^2 + \left[\frac{\partial g}{\partial z}\right]^2} \, dA$$

- **SURFACE AREA OF PORTION OF SURFACE $x = h(y, z)$ [OPTIONAL]:**
  - Let $h(y, z) \in C^{(1,1)}(D^{[yz]})$ in a region $D^{[yz]} \subset \mathbb{R}^2$ of the yz-plane.
    Let S be the portion of the surface $x = h(y, z)$ that lies directly over $D^{[yz]}$.
    (Region $D^{[yz]}$ is also known as the **projection** of S onto the yz-plane.)

    Then:

    $$\text{Surface Area}(S) := \iint_S \, dS = \iint_{D^{[yz]}} \sqrt{1 + [h_y]^2 + [h_z]^2} \, dA = \iint_{D^{[yz]}} \sqrt{1 + \left[\frac{\partial h}{\partial y}\right]^2 + \left[\frac{\partial h}{\partial z}\right]^2} \, dA$$

EX 12.4.1: Setup a double integral for the surface area of $z = e^{-x}\sin y$ over the disk $D = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 5\right\}$.

EX 12.4.2: Using double integral(s), compute the surface area of the plane $2x + 2y - z = 0$ over the triangular region in the xy-plane with vertices $(0, 0, 0)$, $(0, 4, 0)$, $(2, 3, 0)$.

EX 12.4.3: Using double integral(s), compute the surface area of the portion of the paraboloid $z = x^2 + y^2$ that lies inside the sphere $x^2 + y^2 + z^2 = 2$.

EX 12.4.4: Using double integral(s), find a formula for the surface area of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 0$ and $z = h$, where $h > 0$.