TRIPLE INTEGRALS IN RECTANGULAR COORDINATES [SST 12.5]

- **PROPERTIES OF TRIPLE INTEGRALS:**
  - Let set $E \subset \mathbb{R}^3$ be a closed & bounded solid in xyz-space.
    Let functions $f(x,y,z)$ & $g(x,y,z)$ be defined & continuous on E.
    Let $k \in \mathbb{R}$.

    Constant Multiple Rule: $\iiint_E kf\ dV = k \iiint_E f\ dV$

    Sum/Difference Rule: $\iiint_E \left[f \pm g\right] dV = \iiint_E f\ dV \pm \iiint_E g\ dV$

    Nonnegativity Rule: $f(x,y,z) \geq 0\ \ \forall (x,y,z) \in E \implies \iiint_E f\ dV \geq 0$

    Dominance Rule: $f(x,y,z) \leq g(x,y,z)\ \ \forall (x,y,z) \in E \implies \iiint_E f\ dV \leq \iiint_E g\ dV$

    Solid Additivity Rule: $E = E_1 \cup E_2 \implies \iiint_E f\ dV = \iiint_{E_1} f\ dV + \iiint_{E_2} f\ dV$

- **VOLUME OF A (RECTANGULAR) BOX USING TRIPLE INTEGRALS:**
  - $\text{Volume}(E) := \iiint_E dV = \int_a^b \int_c^d \int_p^q dz\ dy\ dx = \int_c^d \int_a^b \int_p^q dz\ dx\ dy = \int_a^b \int_p^q \int_c^d dy\ dz\ dx$
    $= \int_p^q \int_a^b \int_c^d dy\ dx\ dz = \int_c^d \int_p^q \int_a^b dx\ dz\ dy = \int_p^q \int_c^d \int_a^b dx\ dy\ dz$

- **VOLUME OF A z-SIMPLE SOLID USING TRIPLE INTEGRALS:**
  Let $E \subset \mathbb{R}^3$ be a closed & bounded solid in xyz-space.
  Moreover, let E be z-simple with top boundary surface $z = f_2(x,y)$ & bottom boundary surface $z = f_1(x,y)$.
  Finally, let region D be the projection of solid E onto the xy-plane.
  - $\text{Volume}(E) := \iiint_E dV = \iint_D \left[\int_{\text{btm BS in } E}^{\text{top BS in } E} dz\right] dA = \iint_D \left[\int_{f_1(x,y)}^{f_2(x,y)} dz\right] dA$
  - REMARK: BS $\equiv$ **B**oundary **S**urface

- **VOLUME OF A y-SIMPLE SOLID USING TRIPLE INTEGRALS [OPTIONAL]:**
  Let $E \subset \mathbb{R}^3$ be a closed & bounded solid in xyz-space.
  Moreover, let E be y-simple with right boundary surface $y = g_2(x,z)$ & bottom boundary surface $y = g_1(x,z)$.
  Finally, let region D be the projection of solid E onto the xz-plane.
  - $\text{Volume}(E) := \iiint_E dV = \iint_D \left[\int_{\text{left BS in } E}^{\text{right BS in } E} dy\right] dA = \iint_D \left[\int_{g_1(x,z)}^{g_2(x,z)} dy\right] dA$
  - REMARK: BS $\equiv$ **B**oundary **S**urface

- **VOLUME OF A x-SIMPLE SOLID USING TRIPLE INTEGRALS [OPTIONAL]:**
  Let $E \subset \mathbb{R}^3$ be a closed & bounded solid in xyz-space.
  Moreover, let solid E be x-simple with front boundary surface $x = h_2(y,z)$ & back boundary surface $x = h_1(y,z)$.
  Finally, let region D be the projection of solid E onto the yz-plane.
  - $\text{Volume}(E) := \iiint_E dV = \iint_D \left[\int_{\text{back BS in } E}^{\text{front BS in } E} dx\right] dA = \iint_D \left[\int_{h_1(y,z)}^{h_2(y,z)} dx\right] dA$
  - REMARK: BS $\equiv$ **B**oundary **S**urface

EX 12.5.1: Compute $I = \int_0^1 \int_0^2 \int_0^3 \, dx \, dy \, dz$.

EX 12.5.2: Compute $I = \int_0^1 \int_{\sqrt{x}}^{\sqrt{1+x}} \int_0^{xy} \frac{z}{y}\, dz\; dy\; dx.$

EX 12.5.3: Compute $I = \iiint_E 24xy^2z^3 \; dV$, where $E = \{(x, y, z) \in \mathbb{R}^3 : 2 \leq x \leq 3, -1 \leq y \leq 1, 0 \leq z \leq 2\}$.

EX 12.5.4: Compute $I = \iiint_E xy\, dV$, where E is the solid in the first octant bounded by hemisphere $z = \sqrt{9 - x^2 - y^2}$ and the coordinate planes ($x = 0$, $y = 0$, $z = 0$).

EX 12.5.5: Using a triple integral, find the volume of the solid E bounded by paraboloid $z = 9 - 9x^2 - 9y^2$ and plane $z = 3$.

EX 12.5.6: Using a triple integral, find the volume of the solid E bounded above by the surface $x^2 + y^2 + z^3 = 9$ and below by the plane $z = 0$.