TRIPLE INTEGRALS: CYLINDRICAL & SPHERICAL COORDINATES [SST 12.7]

- **SURFACES WITH SIMPLE REPRESENTATION IN CYLINDRICAL COORDINATES:** $(k \in \mathbb{R})$

SURFACE	CYLINDRICAL FORM
z-simple (Circular) Cylinder	$r = k$
z-simple (Circular) Cone	$r = kz$
z-simple (Circular) Paraboloid	$r^2 = kz$
z-simple (Circular) Hyperboloid of 1 Sheet	$r^2 = z^2 + 1$
z-simple (Circular) Hyperboloid of 2 Sheets	$r^2 = z^2 - 1$

- **RECTANGULAR TO CYLINDRICAL COORDINATES:**

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Let $f(x,y,z) \in C(E)$, where solid $E \subset \mathbb{R}^3$ is z-simple s.t. its proj. D is r-simple.

$$\iiint_E f\ dV \overset{CYL}{=} \int_{\text{Smallest }\theta\text{-value in }D}^{\text{Largest }\theta\text{-value in }D} \int_{\text{Inner BC of }D}^{\text{Outer BC of }D} \int_{\text{Btm BS in cyl. form}}^{\text{Top BS in cyl. form}} f\ r\ dz\ dr\ d\theta$$
$$= \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r\cos\theta, r\sin\theta)}^{f_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta$$

REMARK: If region D only has an outer BC, then the inner BC is the **pole** $(r = 0)$.

REMARK: **Always** integrate in the order $dz\ dr\ d\theta$.

- **SURFACES WITH SIMPLE REPRESENTATION IN SPHERICAL COORDINATES:** $(k \in \mathbb{R})$

SURFACE	SPHERICAL FORM	REMARKS (IF ANY)
Sphere	$\rho = k$	$k > 0$
z-simple (Circular) Half-Cone	$\phi = k$	$k \neq \pi/2$
xy-plane	$\phi = \pi/2$	
Plane $z = k$	$\rho = k\sec\phi$	
Plane $y = k$	$\rho = k\csc\phi\csc\theta$	
Plane $x = k$	$\rho = k\csc\phi\sec\theta$	

- **RECTANGULAR TO SPHERICAL COORDINATES:** $\left(0 \leq \phi \leq \pi\right)$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Let $f(x,y,z) \in C(E)$ s.t. $E \subset \mathbb{R}^3$ is a **closed** & **bounded** solid . Then:

$$\iiint_E f\ dV \overset{SPH}{=} \int_{\text{Smallest }\theta\text{-val in }E}^{\text{Largest }\theta\text{-val in }E} \int_{\text{Smallest }\phi\text{-val in }E}^{\text{Largest }\phi\text{-val in }E} \int_{\text{Inside BS of }E}^{\text{Outside BS of }E} f\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

— OR EQUVIALENTLY —

$$\iiint_E f\ dV \overset{SPH}{=} \int_{\text{Smallest }\phi\text{-val in }E}^{\text{Largest }\phi\text{-val in }E} \int_{\text{Smallest }\theta\text{-val in }E}^{\text{Largest }\theta\text{-val in }E} \int_{\text{Inside BS of }E}^{\text{Outside BS of }E} f\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$= \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

REMARK: Setting up iterated triple integral is harder since projection of E on xy-plane is **useless**!

REMARK: **Always** integral in either $d\rho\ d\phi\ d\theta$ or $d\rho\ d\theta\ d\phi$ order.

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EX 12.7.1: Using cylindrical coordinates, compute $I = \iiint_E \sqrt[3]{x^2 + y^2}\, dV$, where E is the solid half-cylinder $\left\{ \begin{array}{c} x^2 + y^2 \le 1 \\ y \ge 0 \\ 1 \le z \le 3 \end{array} \right\}$.

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EX 12.7.2: Using cylindrical coordinates, compute $I = \iiint_E z^2 \, dV$,

where E is the solid bounded by the cylinders $\left\{ \begin{array}{c} x^2 + y^2 = 1 \\ 0 \leq z \leq 2 \end{array} \right\}$ & $\left\{ \begin{array}{c} x^2 + y^2 = 4 \\ 0 \leq z \leq 2 \end{array} \right\}$.

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EX 12.7.3: Using cylindrical coordinates, compute $I = \iiint_E dV$, where E is the solid bounded above by plane $z = 3$ & below by paraboloid $2z = x^2 + y^2$.

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EX 12.7.4: Using spherical coordinates, compute $I = \iiint_E \frac{1}{\sqrt{x^2+y^2+z^2}}\, dV$,

where E is the right solid hemisphere $\left\{ \begin{array}{c} x^2+y^2+z^2 \le 1 \\ y \ge 0 \end{array} \right\}$.

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EX 12.7.5: Using spherical coordinates, compute $I = \iiint_E z \, dV$, where E is the solid bounded above by plane $z = 3$ & below by the half-cone $z = \sqrt{x^2 + y^2}$.

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