MULTIPLE INTEGRALS: CHANGE OF COORDINATES [SST 12.8]

- **DOUBLE INTEGRALS (CHANGING COORDINATES):**
  - Let $D \subset \mathbb{R}^2$ be a closed & bounded region in the xy-plane.
    Let $D^* \subset \mathbb{R}^2$ be a closed & bounded region in the uv-plane.
    Let $f \in C(D)$.
    Let transformation T map region D to region D^* s.t. $T: \begin{cases} x = T_1(u,v) \\ y = T_2(u,v) \end{cases}$ where $T_1, T_2 \in C^{(1,1)}(D^*)$.
    Then $$\boxed{\iint_D f(x,y)\, dA = \iint_{D^*} f\left[T_1(u,v), T_2(u,v)\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, dA}$$
    where $\dfrac{\partial(x,y)}{\partial(u,v)} := \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ is called the **Jacobian**.
  - Useful in simplifying the integrand and/or simplifying the region of integration.
  - Given inverse transformation $T^{-1}: \begin{cases} u = T_1^{-1}(x,y) \\ v = T_2^{-1}(x,y) \end{cases}$ where $T_1^{-1}, T_2^{-1} \in C^{(1,1)}(D)$, then $\dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(x,y)} = 1$

- **TRIPLE INTEGRALS (CHANGING COORDINATES):**
  - Let $E \subset \mathbb{R}^3$ be a closed & bounded solid in xyz-space.
    Let $E^* \subset \mathbb{R}^3$ be a closed & bounded solid in uvw-space.
    Let $f \in C(E)$.
    Let transformation T map solid E to solid E^* s.t. $T: \begin{cases} x = T_1(u,v,w) \\ y = T_2(u,v,w) \\ z = T_3(u,v,w) \end{cases}$ where $T_1, T_2, T_3 \in C^{(1,1,1)}(E^*)$.
    Then $$\boxed{\iiint_E f(x,y,z)\, dV = \iiint_{E^*} f\left[T_1(u,v,w), T_2(u,v,w), T_3(u,v,w)\right] \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \, dV}$$
    where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} := \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$ is called the **Jacobian**.
  - Useful in simplifying the integrand and/or simplifying the region of integration.
  - Given inverse transformation $T^{-1}: \begin{cases} u = T_1^{-1}(x,y,z) \\ v = T_2^{-1}(x,y,z) \\ w = T_3^{-1}(x,y,z) \end{cases}$ where $T_1^{-1}, T_2^{-1}, T_3^{-1} \in C^{(1,1,1)}(E)$,
    then $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 1$

- **DOUBLE INTEGRALS (CHANGING COORDINATES PROCEDURE):**

SETUP: Let transformation T map region D to region D^* s.t. $T: \begin{cases} x = T_1(u,v) \\ y = T_2(u,v) \end{cases}$ where $T_1, T_2 \in C^{(1,1)}(D^*)$.

1. Sketch region D in xy-plane & label BC's (**labeling BP's is unnecessary if too tedious**.)
2. Apply transformation to each BC of region D to obtain a BC of region D^*
3. Sketch region D^* in uv-plane & label BC's & BP's
4. Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ in terms of u & v.
   - REMARK: It may be easier to solve $\dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(x,y)} = 1$
5. Compute the **absolute value** of Jacobian $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$
6. Finally, $\displaystyle\iint_D f(x,y)\, dA = \iint_{D^*} f\left[T_1(u,v), T_2(u,v)\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, dA$

EX 12.8.1: Use the transformation $T : \begin{cases} x = 2u + v \\ y = 3v \end{cases}$ to compute $\displaystyle\iint_D (x - 2y)\, dy\, dx$, where D is the region enclosed by the parallelogram with vertices $(0, 0)$, $(1, 3)$, $(3, 3)$, $(2, 0)$.

EX 12.8.2: Use the inverse transformation $T^{-1} : \begin{cases} u = x - y \\ v = x + y \end{cases}$ to compute $\displaystyle\iint_D \left(\frac{x-y}{x+y}\right)^8 \, dy \, dx$, where D is the region enclosed by the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$.

EX 12.8.3: Use the transformation $T: \begin{cases} x = u\sqrt{5} \\ y = 2v \end{cases}$ to compute $\iint_D y^2 \, dA$,

where D is the region enclosed by the ellipse $\dfrac{x^2}{5} + \dfrac{y^2}{4} = 1$.

EX 12.8.4: Use the inverse transformation $T^{-1}: \begin{cases} u = xy \\ v = x^2 - y^2 \end{cases}$ to **setup** $\iint_D dA$,

where D is the region in Quadrant I bounded by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$ & $x^2 - y^2 = 4$.

EX 12.8.5: Compute the Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ of the transformation $T : \begin{cases} x = 2uv \\ y = 3vw \\ z = 4uw \end{cases}$.