VECTOR FIELDS: INTRO, DIV, CURL [SST 13.1]

- **THE FUNCTION LANDSCAPE:**

FUNCTION TYPE	PROTOTYPE	MAPPING
(Scalar) Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$
Function of 2 Variables	$z = f(x, y)$	$f : \mathbb{R}^2 \to \mathbb{R}$
Function of 3 Variables	$w = f(x, y, z)$	$f : \mathbb{R}^3 \to \mathbb{R}$

2D Vector Field	$\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$	$\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$
3D Vector Field	$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$	$\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$

  - Going forward, multivariable functions $f(x, y)$, $g(x, y, z)$ may be referred to as **scalar fields**.
  - As with vectors, vector fields have similar definitions of scalar multiplication, addition, dot product & cross product.

- **2D VECTOR FIELDS:** $\Big($Let 2D vector field $\mathbf{F}(x, y) \in C^{(1,1)}$ s.t. $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$.$\Big)$
  - (Del Operator) $\nabla := \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} \right\rangle$
  - (Divergence) $\text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} := \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} = M_x + N_y$
  - (Curl) $\text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F} := \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left\langle 0, 0, \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right\rangle$

- **3D VECTOR FIELDS:** $\Big($Let 3D vector field $\mathbf{F}(x, y, z) \in C^{(1,1,1)}$ s.t. $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$.$\Big)$
  - (Del Operator) $\nabla := \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$
  - (Divergence) $\text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} := \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z} = M_x + N_y + P_z$
  - (Curl) $\text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F} := \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left\langle \dfrac{\partial P}{\partial y} - \dfrac{\partial N}{\partial z}, \dfrac{\partial M}{\partial z} - \dfrac{\partial P}{\partial x}, \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \right\rangle$

- **LAPLACIAN:**
  - Given 2D scalar field $f(x, y) \in C^{(2,2)}$

$$\nabla^2 f := \text{div } \Big(\text{grad } f\Big) \equiv \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$$

  - Given 3D scalar field $f(x, y, z) \in C^{(2,2,2)}$

$$\nabla^2 f := \text{div } \Big(\text{grad } f\Big) \equiv \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}$$

  - A scalar field f is **harmonic** if $\nabla^2 f = 0$.

- **VECTOR CALCULUS PROPERTIES & IDENTITIES:**

$\nabla \cdot (k\mathbf{F}) = k\nabla \cdot \mathbf{F}$	$\nabla \times (k\mathbf{F}) = k\nabla \times \mathbf{F}$
$\nabla \cdot (\mathbf{F} \pm \mathbf{G}) = \nabla \cdot \mathbf{F} \pm \nabla \cdot \mathbf{G}$	$\nabla \times (\mathbf{F} \pm \mathbf{G}) = \nabla \times \mathbf{F} \pm \nabla \times \mathbf{G}$
$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + (\nabla f \cdot \mathbf{F})$	$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f \times \mathbf{F})$
$\nabla \cdot (\nabla \times \mathbf{F}) = 0$	$\nabla \times (\nabla f) = \vec{\mathbf{0}}$

$\nabla \cdot (f\nabla g) = f\nabla \cdot (\nabla g) + \nabla f \cdot \nabla g$

$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

$\nabla^2 (fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$

EX 13.1.1: Given vector field $\vec{\mathbf{F}}(x,y) = x\widehat{\mathbf{i}} + \left(y^2\right)\widehat{\mathbf{j}} = \langle x, y^2\rangle$, compute:

(a) $\vec{\mathbf{F}}(1,2)$ (b) $5\vec{\mathbf{F}}$ (c) $xy\vec{\mathbf{F}}$ (d) $\nabla\cdot\vec{\mathbf{F}}$ (e) $\nabla\cdot\vec{\mathbf{F}}(1,2)$

EX 13.1.2: Given vector fields $\vec{\mathbf{F}}(x, y, z) = \langle x, y^2, z^3 \rangle$ and $\vec{\mathbf{G}}(x, y, z) = \langle xy, yz, xz \rangle$, compute:

(a) $\vec{\mathbf{G}}(1, 2, -3)$ (b) $\vec{\mathbf{F}} \cdot \vec{\mathbf{G}}$ (c) $\vec{\mathbf{F}} \times \vec{\mathbf{G}}$ (d) $\nabla \times \vec{\mathbf{G}}$ (e) $\nabla \times \vec{\mathbf{G}}(-2, 0, 1)$

EX 13.1.3: Given scalar fields $f(x, y) = \cos x \sin y$ and $g(x, y, z) = e^{xyz}$, compute:

(a) $\nabla^2 f$ (b) $\nabla^2 f(\pi/3, \pi/3)$ (c) $\nabla^2 g$ (d) $\nabla^2 g(1, 0, 0)$

EX 13.1.4: Given vector field $\vec{\mathbf{F}} = \nabla f$, where $f(x, y, z) = xy^2z^3$, compute $\nabla \cdot \vec{\mathbf{F}}$.

EX 13.1.5: Given vector field $\vec{\mathbf{H}}(x, y, z) = xyz\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + z\widehat{\mathbf{k}} = \langle xyz, y, z\rangle$, compute:

(a) $\nabla \times \vec{\mathbf{H}}$ (b) $\nabla \cdot (\nabla \times \vec{\mathbf{H}})$ (c) $\nabla \times (\nabla \times \vec{\mathbf{H}})$

EX 13.1.6: Is $f(x,y,z) = \sqrt{1+x^2+y^2+z^2}$ harmonic in $\mathbb{R}^3$? (Justify answer)