LINE INTEGRALS [SST 13.2]

- **LINE INTEGRAL OF A SCALAR FIELD:**

  – Given smooth curve $C: \begin{cases} x = x(t) \\ y = y(t) \\ t \in [a,b] \end{cases}$ and continuous scalar field $f(x,y)$

  Then the **line integral** of f over curve C is defined to be

$$\int_C f(x,y)\ ds := \int_{\text{Starting } t\text{-value in } C}^{\text{Ending } t\text{-value in } C} f\Big[x(t),y(t)\Big]\sqrt{[x'(t)]^2+[y'(t)]^2}\ dt = \int_a^b f\Big[x(t),y(t)\Big]\sqrt{[x'(t)]^2+[y'(t)]^2}\ dt$$

$$\int_C f(x,y)\ dx = \int_{\text{Starting } t\text{-value in } C}^{\text{Ending } t\text{-value in } C} f\Big[x(t),y(t)\Big]\ x'(t)\ dt = \int_a^b f\Big[x(t),y(t)\Big]\ x'(t)\ dt$$

$$\int_C f(x,y)\ dy = \int_{\text{Starting } t\text{-value in } C}^{\text{Ending } t\text{-value in } C} f\Big[x(t),y(t)\Big]\ y'(t)\ dt = \int_a^b f\Big[x(t),y(t)\Big]\ y'(t)\ dt$$

  – Given smooth curve $\Gamma: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \\ t \in [a,b] \end{cases}$ and continuous scalar field $f(x,y,z)$.

  Then the **line integral** of f over curve Γ is defined to be

$$\int_\Gamma f(x,y,z)\ ds := \int_a^b f\Big[x(t),y(t),z(t)\Big]\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\ dt$$

$$\int_\Gamma f(x,y,z)\ dx = \int_a^b f\Big[x(t),y(t),z(t)\Big]\ x'(t)\ dt$$

$$\int_\Gamma f(x,y,z)\ dy = \int_a^b f\Big[x(t),y(t),z(t)\Big]\ y'(t)\ dt$$

$$\int_\Gamma f(x,y,z)\ dz = \int_a^b f\Big[x(t),y(t),z(t)\Big]\ z'(t)\ dt$$

- **LINE INTEGRAL OF A VECTOR FIELD:**

  – Let vector function $\vec{\mathbf{R}}(t) = \langle x(t), y(t)\rangle$ trace out a smooth curve C in $\mathbb{R}^2$.
  Let vector field $\vec{\mathbf{F}}(x,y) = \langle u(x,y), v(x,y)\rangle$ be continuous on C.

  Then the **line integral of $\vec{\mathbf{F}}$ along** C is defined by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} := \int_C \Big[u(x,y)\ dx + v(x,y)\ dy\Big]$$

  where $d\vec{\mathbf{R}} = \langle dx, dy\rangle$

  – Let vector function $\vec{\mathbf{R}}(t) = \langle x(t), y(t), z(t)\rangle$ trace out a smooth curve Γ in $\mathbb{R}^3$.
  Let vector field $\vec{\mathbf{F}}(x,y,z) = \langle u(x,y,z), v(x,y,z), w(x,y,z)\rangle$ be continuous on Γ.

  Then the **line integral of $\vec{\mathbf{F}}$ along** Γ is defined by

$$\int_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} := \int_\Gamma \Big[u(x,y,z)\ dx + v(x,y,z)\ dy + w(x,y,z)\ dz\Big]$$

  where $d\vec{\mathbf{R}} = \langle dx, dy, dz\rangle$

- **MASS OF A THIN WIRE:** Mass $= \int_C \rho(x,y,z)\ ds$, where wire is in the shape of curve C and has density $\rho(x,y,z)$.

- **WORK AS A LINE INTEGRAL:** Work $= \int_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$, where $\vec{\mathbf{F}}$ is a **force field** and object is moved along curve Γ.

EX 13.2.1: Compute the line integral $\int_C \frac{1}{x+1}\,ds$, where curve $C : \begin{cases} x = 2t \\ y = t \\ 0 \le t \le 1 \end{cases}$

EX 13.2.2: Compute the line integral $\int_\Gamma \frac{y^2}{x^3}\, ds$, where curve $\Gamma : \begin{cases} x = 2t \\ y = t^4 \\ t \in [0, 1] \end{cases}$

EX 13.2.3: Compute $\int_C \left(-xy^2 \; dx + x^2 \; dy \right)$, where C is the path from $(-1, 1)$ to $(0, 1)$ along line $y = 1$ and then from $(0, 1)$ to $(1, 0)$ along quarter-circle $x^2 + y^2 = 1$.

EX 13.2.4: Compute $\int_\Gamma \Big(-y\ dx + x\ dy + xz\ dz \Big)$, where curve $\Gamma : \begin{cases} x = \cos t \\ y = \sin t \\ z = t \\ t \in [0, 2\pi] \end{cases}$

EX 13.2.5: Compute $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$, where $\vec{\mathbf{F}}(x, y) = \langle 5x + y, x \rangle$ and C is the curve traced by $\vec{\mathbf{R}}(t) = \langle t^2, -t \rangle$ for $0 \leq t \leq 1$.

EX 13.2.6: Find the work done by the force field $\vec{\mathbf{F}}(x, y, z) = \langle 2xy, x^2 + 2, y \rangle$ on an object moving along Γ, where Γ is the line segment from $(1, 0, 2)$ to $(3, 4, 1)$.

EX 13.2.7: Find the total mass of a thin wire with density $\rho(x, y, z) = 3xyz$ in the shape of curve $C : \begin{cases} x = 2t \\ y = t^2 \\ z = \frac{1}{3}t^3 \\ t \in [0, 2] \end{cases}$