GRADIENT FIELDS, SCALAR POTENTIALS, PATH INDEPENDENCE [SST 13.3]

- **GRADIENT FIELD (CONSERVATIVE VECTOR FIELD):**

  $\vec{\mathbf{F}}$ is a **gradient field** $\iff$ $\vec{\mathbf{F}}$ is **conservative** $\iff$ $\vec{\mathbf{F}} = \nabla f$ for some scalar field f $\iff$ f is a **scalar potential** for $\vec{\mathbf{F}}$

- **FUNDAMENTAL THEOREM OF CALCULUS (FTC):**

  Let $[a, b]$ be a closed interval traced out by x. In other words, $x \in [a, b]$.

  Let scalar function $f \in C^1[a, b]$. Then

$$\int_a^b f'(x)\, dx \overset{FTC}{=} f(b) - f(a)$$

- **FUNDAMENTAL THEOREM FOR LINE INTEGRALS (FTLI):**

  Let Γ be a piecewise smooth curve traced out by $\vec{\mathbf{R}}(t)$ for $t \in [a, b]$.

  Let scalar field $f \in C^1(\Gamma)$. Then

$$\int_\Gamma \nabla f \cdot d\vec{\mathbf{R}} \overset{FTLI}{=} f\left[\vec{\mathbf{R}}(b)\right] - f\left[\vec{\mathbf{R}}(a)\right]$$

- **CROSS-PARTIALS TEST FOR A GRADIENT FIELD IN $\mathbb{R}^2$:**

  Let $D \in \mathbb{R}^2$ be a simply-connected region on the xy-plane.

  Let vector field $\vec{\mathbf{F}} \in C^{(1,1)}(D)$ s.t. $\vec{\mathbf{F}}(x, y) = \langle M(x, y), N(x, y)\rangle$. Then:

$$\vec{\mathbf{F}} \text{ is conservative} \iff \vec{\mathbf{F}} \text{ is a gradient field} \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- **CURL TEST FOR A GRADIENT FIELD IN $\mathbb{R}^3$:**

  Let $E \in \mathbb{R}^3$ be a simply-connected solid in xyz-space.

  Let vector field $\vec{\mathbf{F}} \in C^{(1,1,1)}(E)$ s.t. $\vec{\mathbf{F}}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$. Then:

$$\vec{\mathbf{F}} \text{ is conservative} \iff \vec{\mathbf{F}} \text{ is a gradient field} \iff \nabla \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

- **PATH INDEPENDENCE:**

  Let S be an open connected set in either $\mathbb{R}^2$ or $\mathbb{R}^3$.

  Then line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$ is **independent of path** in S if for any two points $P, Q \in S$, the line integral along every piecewise smooth curve in S from P to Q has the same value.

- **EQUIVALENT CONDITIONS FOR PATH INDEPENDENCE:**

  Let S be an open connected set in either $\mathbb{R}^2$ or $\mathbb{R}^3$.

  Let vector field $\vec{\mathbf{F}}$ be continous on S.

  Then the following are equivalent (TFAE):

  (i) $\vec{\mathbf{F}}$ is a gradient field on S. i.e. $\vec{\mathbf{F}} = \nabla f$ for some **scalar potential** f.

  (ii) $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = 0$ for every piecewise smooth **closed** curve C in S.

  (iii) $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$ is independent of path within S.

- **SIMPLER PATH CHOICES FOR PATH INDEPENDENT LINE INTEGRALS:**

  Suppose the path Γ for $\int_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$ starts at point $P(x_0, y_0)$, ends at point $Q(x_1, y_1)$, and is too "complicated."

  Moreover, suppose finding a **scalar potential** for $\vec{\mathbf{F}}$ is too tedious or not possible (due to a **nonelementary integral**.)

  - A simpler path Γ^* is the **straight line path** from P to Q: $\vec{\mathbf{R}}^*(t) := \langle x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t\rangle$ for $t \in [0, 1]$.
  - A simpler path Γ^* is the **union of one horizontal path & one vertical path** from P to Q: $\Gamma^* = \Gamma^*_{HL} \cup \Gamma^*_{VL}$ where

    $\vec{\mathbf{R}}^*_{HL}(t) := \langle x_0 + (x_1 - x_0)t, y_0\rangle$ for $t \in [0, 1]$ and $\vec{\mathbf{R}}^*_{VL}(t) := \langle x_1, y_0 + (y_1 - y_0)t\rangle$ for $t \in [0, 1]$

    —- OR —-

    $\vec{\mathbf{R}}^*_{VL}(t) := \langle x_0, y_0 + (y_1 - y_0)t\rangle$ for $t \in [0, 1]$ and $\vec{\mathbf{R}}^*_{HL}(t) := \langle x_0 + (x_1 - x_0)t, y_1\rangle$ for $t \in [0, 1]$

  Corresponding simpler paths apply for three dimensions.

EX 13.3.1: Let vector field $\vec{\mathbf{F}}(x,y) = \langle y^2, 2xy - 3\rangle$ and C be any smooth path from $P(0,0)$ to $Q(2,-1)$.

(a) Verify that $\vec{\mathbf{F}}$ is conservative. (b) Find a scalar potential f for $\vec{\mathbf{F}}$. (c) Compute $I = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$.

EX 13.3.2: Let vector field $\vec{\mathbf{F}}(x, y, z) = \langle yze^{xy}, xze^{xy}, e^{xy} \rangle$ and Γ be any smooth path from $P(7, 0, 2)$ to $Q(0, 8, 12)$.

(a) Verify that $\vec{\mathbf{F}}$ is a gradient field.

(b) Find a scalar potential f for $\vec{\mathbf{F}}$.

(c) Compute $I = \int_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$.

EX 13.3.3: Let C be the path traced by $\vec{\mathbf{R}}(t) = \langle 2\sin\left(\frac{\pi t}{2}\right)\cos(\pi t), \arcsin t\rangle$ for $0 \le t \le 1$.

(a) Verify that line integral $I = \int_C \left[(\sin y)\ dx + (3 + x\cos y)\ dy\right]$ is independent of path (IoP).

(b) Compute line integral I **without finding a scalar potential**.

<u>EX 13.3.4:</u> Let Γ be the path traced by $\vec{\mathbf{R}}(t) = \langle t, t^2, t^4 \rangle$ for $0 \le t \le 1$.

(a) Verify that line integral $I = \int_\Gamma \left[(4xe^z)\ dx + (\cos y)\ dy + (2x^2 e^z)\ dz \right]$ is independent of path (IoP).

(b) Compute line integral I **<u>without</u> finding a scalar potential**.