SURFACE INTEGRALS & FLUX INTEGRALS [SST 13.5]

- **SMOOTH SURFACE:**

  A surface S is **smooth** if the normal vector at each point in S exists & is nonzero.

- **PIECEWISE SMOOTH SURFACE:**

  A surface S is **piecewise smooth** if it is the union of a finite # of smooth subsurfaces.

- **SURFACE INTEGRAL:**

  Let surface $S \subset \mathbb{R}^3$ be defined by $z = f(x, y)$.

  Let region $D \subset \mathbb{R}^2$ be the projection of surface S onto the xy-plane.

  Let $f \in C^{(1,1)}(D)$ and $g \in C(S)$.

  Then the **surface integral** of g **across** S is defined to be:

$$\underbrace{\iint_S g(x,y,z)\ dS}_{surface\ integral} := \underbrace{\iint_D g(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\ dA}_{double\ integral}$$

  REMARK: $\displaystyle\iint_S dS$ gives the **surface area** of the portion of S over region D in xy-plane.

- **ORIENTABLE SURFACE:**

  A surface S is **orientable** if S has a unit normal vector field $\widehat{\mathbf{N}}(x, y, z) \in C(S)$.

  Geometrically, an orientable surface is "2-sided."

  REMARK: Most typical surfaces are orientable.

  REMARK: Examples of non-orientable surfaces are: Möbius Strip, Klein Bottle

- **FLUX INTEGRAL:**

  Let orientable surface $S \subset \mathbb{R}^3$ be described by $z = f(x, y)$ w/ unit normal field $\widehat{\mathbf{N}}$.

  Let region $D \subset \mathbb{R}^2$ be the projection of surface S onto the xy-plane.

  Let vector field $\vec{\mathbf{F}}(x, y, z) \in C^{(1,1,1)}(S)$ and scalar field $f \in C^{(1,1)}(D)$.

  Then the **flux (integral) of $\vec{\mathbf{F}}$ across** S is:

$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\ dS = \iint_D \vec{\mathbf{F}}\left(x, y, f(x,y)\right) \cdot \langle -f_x, -f_y, 1\rangle\ dA \qquad \text{(if } \widehat{\mathbf{N}} \text{ is upward)}$$

$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\ dS = \iint_D \vec{\mathbf{F}}\left(x, y, f(x,y)\right) \cdot \langle f_x, f_y, -1\rangle\ dA \qquad \text{(if } \widehat{\mathbf{N}} \text{ is downward)}$$

EX 13.5.1: Let surface S be the portion of the plane $z = 4 - x - y$ above the xy-plane for which $x \geq 0$ and $y \geq 0$.

Compute the surface integral $I = \iint_S xy \, dS$.

EX 13.5.2: Let surface S be the portion of the plane $z = 4$ for which $x^2 + y^2 \leq 1$.

Compute the surface integral $I = \iint_S \left(x^2 + y^2\right)\, dS$.

EX 13.5.3: Let surface S be the portion of the paraboloid $z = x^2 + y^2$ for which $z \leq 4$.

Compute the surface integral $I = \iint_S \frac{1}{\sqrt{1+4z}}\, dS$.

EX 13.5.4: Let surface S be the portion of the hemisphere $x^2 + y^2 + z^2 = 5$ for which $z \geq 1$, oriented upward.

Let vector field $\vec{\mathbf{F}}(x, y, z) = \langle 2x, 2y, 0 \rangle$.

Compute the flux integral $I = \iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}} \, dS$.