VECTORS: DOT PRODUCT [SST 9.3]

DOT PRODUCT OF TWO VECTORS:

- In $\mathbb{R}^2$, $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle \implies$ **Dot product** $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$
- In $\mathbb{R}^3$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle \implies$ **Dot product** $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
- Notice that the dot product of two vectors is a scalar!

PROPERTIES OF DOT PRODUCTS:

$\left(\text{Let vectors } \mathbf{u}, \mathbf{v}, \mathbf{w} \in \left\{\mathbb{R}^2, \mathbb{R}^3\right\} \text{ and scalar } c \in \mathbb{R}\right)$

- $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$
- $\vec{\mathbf{0}} \cdot \mathbf{v} = \mathbf{v} \cdot \vec{\mathbf{0}} = 0$
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

ANGLE BETWEEN VECTORS:

- $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}|| \cos(\theta_{vw})$, where the **angle between the two vectors** $\theta_{vw} \in [0, \pi]$ and $\mathbf{v}, \mathbf{w} \in \left\{\mathbb{R}^2, \mathbb{R}^3\right\}$
- Nonzero vectors $\mathbf{v}, \mathbf{w}$ are **orthogonal** $\iff \mathbf{v} \cdot \mathbf{w} = 0$
- Nonzero vectors $\mathbf{v}, \mathbf{w}$ are **parallel** $\iff \mathbf{v} = t\mathbf{w}$ for some scalar $t \in \mathbb{R}$.

PROJECTIONS:

- **Vector projection of v** onto **w**: $\text{proj}_{\mathbf{w}} \mathbf{v} = \left(\dfrac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$
- **Scalar projection of v** onto **w**: $\text{comp}_{\mathbf{w}} \mathbf{v} = \pm\, ||\text{proj}_{\mathbf{w}} \mathbf{v}|| = \dfrac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||}$

WORK AS A DOT PRODUCT:

- The **work** W done by a **constant force F** on an object moving along the line from point P to point Q is: $W = \mathbf{F} \cdot \mathbf{PQ}$

EX 9.3.1: (a) Find the dot product $\mathbf{v} \cdot \mathbf{w}$ if $\mathbf{v} = \langle 1, 2 \rangle$ and $\mathbf{w} = 4\widehat{\mathbf{i}} - 2\widehat{\mathbf{j}}$. (b) Are $\mathbf{v}, \mathbf{w}$ orthogonal?

(c) Find the angle θ_{vw} between $\mathbf{v}$ & $\mathbf{w}$.

EX 9.3.2: (a) Find the dot product $\mathbf{v} \cdot \mathbf{w}$ if $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = 4\widehat{\mathbf{i}} - 5\widehat{\mathbf{j}} - 6\widehat{\mathbf{k}}$. (b) Are $\mathbf{v}, \mathbf{w}$ orthogonal?

(c) Find the angle θ between $\mathbf{v}$ & $\mathbf{w}$. (d) Are $\mathbf{v}, \mathbf{w}$ parallel?

EX 9.3.3: Find scalar t such that vectors $\mathbf{v} = \langle 1, 3, t \rangle$ and $\mathbf{w} = 2t\widehat{\mathbf{i}} + \widehat{\mathbf{j}} - 4\widehat{\mathbf{k}}$ are orthogonal.

EX 9.3.4: Find the (vector) projection of $\mathbf{v} = \langle 1, 2 \rangle$ onto $\mathbf{w} = -3\widehat{\mathbf{i}} + 4\widehat{\mathbf{j}}$.

EX 9.3.5: Find the (vector) projection of $\mathbf{v} = \widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + 3\widehat{\mathbf{k}}$ onto $\mathbf{w} = \langle 3, -1, -1 \rangle$.

EX 9.3.6: Find the work performed when the force $\mathbf{F} = \langle 1, 2, 3 \rangle$ moves an object along a straight line from point $P(-2, 3, 4)$ to point $Q(8, -7, -6)$.