PARAMETRIC CURVES [SST 9.5]

- **EXPLICIT RECTANGULAR CURVES IN $\mathbb{R}^2$:**
  - Form: $y = f(x)$ OR $x = g(y)$
  - Examples: $y = 1 + x^4$, $y = \tan(3x)$, $x = \sqrt[3]{2y-1}$, $x = -\log y$, ...
  - REMARK: Such curves never touch or cross themselves.
- **IMPLICIT RECTANGULAR CURVES IN $\mathbb{R}^2$:**
  - Form: $F(x, y) = k$, where $k \in \mathbb{R}$.
  - Examples: $x^2 - 3y^2 = 5$, $\sin(xy) + xy = 1$, $(x^2 + y^2)^2 - 4x^2y = 0$, ...
  - REMARK: Often, rewriting an implicit form to explicit form is impossible.
- **EXPLICIT POLAR CURVES IN $\mathbb{R}^2$:**
  - Form: $r = f(\theta)$ OR $\theta = k$, where $k \in \mathbb{R}$.
  - Examples: $r = 3$, $r = e^\theta$, $r = 1 - 4\sin\theta$, $\theta = \pi/3$, ...
- **IMPLICIT POLAR CURVES IN $\mathbb{R}^2$:**
  - Form: $F(r, \theta) = k$, where $k \in \mathbb{R}$.
  - Examples: $e^{r\theta} = 1$, $r^2 = 4\sin\theta$, $r + r^2\sin(3\theta) = 0$, ...

- **PARAMETRIC CURVES IN $\mathbb{R}^2$:**
  - Form: $\begin{cases} x = f(t) \\ y = g(t) \\ t \in I \end{cases}$, where I is an **interval**, $f, g \in C(I)$
  - Idea: Each point (x, y) on the curve (on the xy-plane) depends on a **parameter**, $t \in \mathbb{R}$.
  - DEFINITION: A particular choice of f, g and I is called a **parameterization** of the curve.
  - REMARK: There exist curves for which no simple parameterization can be determined.
  - REMARK: Parameterizations are not unique!
- **PARAMETRIC CURVES IN $\mathbb{R}^3$:**
  - Form: $\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \\ t \in I \end{cases}$, where I is an **interval**, $f, g, h \in C(I)$
  - Idea: Each point (x, y, z) on the curve (in xyz-space) depends on a **parameter**, $t \in \mathbb{R}$.
  - Similar definition & remarks apply in $\mathbb{R}^3$ as they did in $\mathbb{R}^2$.
  - 3D parametric lines will be considered here, other 3D parametric curves will be seen in Chapter 10.
- **TYPICAL NOTATION FOR PARAMETERS:** $s, t, \lambda, \mu, \theta, \phi$

CONVERSION	DIFFICULTY	PROCEDURE
2D Parametric → Rectangular	Depends on solvability of t	Solve for t in one eqn and substitute into the other eqn. Restrict the ranges of x & y if necessary.
Explicit Rectangular → 2D Parametric	Always works	$y = f(x) \implies \begin{cases} x = t \\ y = f(t) \\ t \in \text{Dom}(f) \end{cases}$ $x = g(y) \implies \begin{cases} x = g(t) \\ y = t \\ t \in \text{Dom}(g) \end{cases}$
Explicit Polar → 2D Parametric	Always works	$r = f(\theta) \implies \begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \\ \theta \in \text{Dom}(f) \end{cases}$

***** ALL OTHER CONVERSION POSSIBILITIES ARE TOO DIFFICULT, SO NOT CONSIDERED HERE *****

LINES IN SPACE [SST 9.5]

- **LINES IN $\mathbb{R}^2$:**
  - The equation of a line as presented in algebra (e.g. $y = mx + b$, ...) is inadequate for 3D.
  - Parametric Form of Line ℓ in $\mathbb{R}^2$: $\begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ t \in \mathbb{R} \end{cases}$, where Line ℓ || vector $\mathbf{v} := \langle v_1, v_2 \rangle$, Line ℓ contains point $P_0(x_0, y_0)$
  - If given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, form vector $\mathbf{v} = \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$

- **LINES IN $\mathbb{R}^3$:**
  - Parametric Form of Line ℓ in $\mathbb{R}^3$: $\begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ z = z_0 + v_3 t \\ t \in \mathbb{R} \end{cases}$, where Line ℓ || vector $\mathbf{v} := \langle v_1, v_2, v_3 \rangle$, Line ℓ contains point $P_0(x_0, y_0, z_0)$
  - If given two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, form vector $\mathbf{v} = \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$
  - Symmetric Form of Line ℓ in $\mathbb{R}^3$: $\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2} = \dfrac{z - z_0}{v_3}$
    - Do not use the symmetric form since it can't handle vectors parallel to an axis: $\langle 0, v_2, v_3 \rangle$, $\langle v_1, 0, v_3 \rangle$, $\langle v_1, v_2, 0 \rangle$
    - Therefore, convert symmetric form to parametric form.

- **PROPERTIES OF LINES IN $\mathbb{R}^3$:** Given lines $\ell_1 : \begin{cases} x = x_1 + v_1 t \\ y = y_1 + v_2 t \\ z = z_1 + v_3 t \\ t \in \mathbb{R} \end{cases}$ and $\ell_2 : \begin{cases} x = x_2 + w_1 s \\ y = y_2 + w_2 s \\ z = z_2 + w_3 s \\ s \in \mathbb{R} \end{cases}$

  ⋆ Lines ℓ_1, ℓ_2 are **perpendicular** $\iff$ $\mathbf{v} \cdot \mathbf{w} = 0$

  ⋆ Lines ℓ_1, ℓ_2 are **parallel** $\iff$ $\mathbf{v} \parallel \mathbf{w}$ $\iff$ $\mathbf{v} = k\mathbf{w}$, for some $k \in \mathbb{R}$

  ⋆ Lines ℓ_1, ℓ_2 are **intersecting** $\iff$ linear system $\begin{cases} x_1 + v_1 t = x_2 + w_1 s \\ y_1 + v_2 t = y_2 + w_2 s \\ z_1 + v_3 t = z_2 + w_3 s \end{cases}$ has a **unique solution** for (t, s).

  ⋆ Lines ℓ_1, ℓ_2 are **coincident** $\iff$ linear system $\begin{cases} x_1 + v_1 t = x_2 + w_1 s \\ y_1 + v_2 t = y_2 + w_2 s \\ z_1 + v_3 t = z_2 + w_3 s \end{cases}$ has **infinitely many solutions** for (t, s).

  ⋆ Lines ℓ_1, ℓ_2 are **skew** $\iff$ ℓ_1, ℓ_2 do not intersect and are not parallel.

- **DISTANCE FROM A POINT TO A LINE:** Given point P and line ℓ

  1. Construct vector $\mathbf{v}$ parallel to line ℓ.
  2. Pick any point Q on the line ℓ.
  3. Form vector $\mathbf{QP}$.
  4. $\Big(\text{Distance from point } P \text{ to line } \ell\Big) = \dfrac{||\mathbf{v} \times \mathbf{QP}||}{||\mathbf{v}||}$

EX 9.5.1: Convert the parametric curve $\begin{cases} x = 3t - 7 \\ y = 2t + 1 \\ t \in \mathbb{R} \end{cases}$ to rectangular form.

EX 9.5.2: Convert the parametric curve $\begin{cases} x = \csc t \\ y = 2\sin t \\ t \in (0, \pi/2) \end{cases}$ to rectangular form.

EX 9.5.3: Convert the parametric curve $\begin{cases} x = 3\sec\theta \\ y = 5\tan\theta \\ -\infty < \theta < \infty \end{cases}$ to rectangular form.

EX 9.5.4: Convert the parametric curve $\begin{cases} x = \frac{1}{\sqrt{t+1}} \\ y = \frac{t}{t+1} \\ t > -1 \end{cases}$ to rectangular form.

EX 9.5.5: Convert the rectangular curve $y = x^2 + x + 3$ to parametric form.

EX 9.5.6: Convert the rectangular curve $x = \sqrt{y}$ to parametric form.

EX 9.5.7: Convert the polar curve $r = 1 - 2\sin\theta$ to parametric form.

EX 9.5.8: Find a parametric form for the line ℓ that contains point $P(1, 2, -3)$ and is parallel to vector $\mathbf{v} = 4\widehat{\mathbf{i}} - 3\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}$.

EX 9.5.9: Find a parametric form for the line ℓ that contains point $P(-2, 1, -5)$ and is parallel to line $\begin{cases} x = 1 + t \\ y = 2 - t \\ z = -3 + 7t \\ t \in \mathbb{R} \end{cases}$.

EX 9.5.10: Find a parametric form for the line ℓ that contains points $P(1, 2, -3)$ and $Q(7, 6, 8)$.

EX 9.5.11: Are lines $\ell_1 : \begin{cases} x = -2 + 3t \\ y = -2t \\ z = 1 + 4t \\ t \in \mathbb{R} \end{cases}$ and $\ell_2 : \begin{cases} x = 3 + 5s \\ y = -1 - s \\ z = 4 + 3s \\ s \in \mathbb{R} \end{cases}$ parallel, intersecting, coincident, or skew?

EX 9.5.12: Are lines $\ell_1 : \begin{cases} x = 1 + t \\ y = 6 - 2t \\ z = 10 - 2t \\ t \in \mathbb{R} \end{cases}$ and $\ell_2 : \begin{cases} x = 3 - 2s \\ y = -1 + 4s \\ z = 12 + 4s \\ s \in \mathbb{R} \end{cases}$ parallel, intersecting, coincident, or skew?

EX 9.5.13: Are lines $\ell_1 : \begin{cases} x = 1 + t \\ y = -2 + 3t \\ z = 4 - t \\ t \in \mathbb{R} \end{cases}$ and $\ell_2 : \begin{cases} x = 2s \\ y = 3 + s \\ z = -3 + 4s \\ s \in \mathbb{R} \end{cases}$ parallel, intersecting, coincident, or skew?

EX 9.5.14: Are lines $\ell_1 : \begin{cases} x = 1 + t \\ y = -2 + 3t \\ z = 4 - t \\ t \in \mathbb{R} \end{cases}$ and $\ell_2 : \begin{cases} x = 11 - s \\ y = 28 - 3s \\ z = -6 + s \\ s \in \mathbb{R} \end{cases}$ parallel, intersecting, coincident, or skew?

EX 9.5.15: Find the distance from the point $P(-2, 4, 6)$ to the line $\ell : \begin{cases} x = 1 + t \\ y = -2 + 3t \\ z = 4 - t \\ t \in \mathbb{R} \end{cases}$