SOLID ANALYTIC GEOMETRY: PLANES [SST 9.6]

- **EQUATIONS OF PLANES:**
  - **Standard Form of a Plane $\mathbb{P}$:** $Ax + By + Cz + D = 0$, where $A, B, C, D \in \mathbb{R}$
  - CASE I: Given the normal vector $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$ and one point $P_0(x_0, y_0, z_0)$:
    1. Let components of normal vector $\langle n_1, n_2, n_3 \rangle$ be the coefficients of x, y, z in standard form of plane.
    2. Substitute the given point (x_0, y_0, z_0) into x, y, z respectively.
    3. Solve the resulting equation for D.
  - CASE II: Given three points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, and $R(x_3, y_3, z_3)$:
    1. Form vector $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$
    2. Form vector $\mathbf{PR} = \langle x_3 - x_1, y_3 - y_1, z_3 - z_1 \rangle$
    3. Form normal vector $\mathbf{n}$ using a cross product: $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR}$
    4. Using normal vector $\mathbf{n}$ and one of the three given points P, Q, R, follow CASE I.
  - CASE III: Given two intersecting lines ℓ_1, ℓ_2:
    1. Form vectors $\mathbf{v}$, $\mathbf{w}$ parallel to lines ℓ_1, ℓ_2 respectively.
    2. Form normal vector to the plane: $\mathbf{n} = \mathbf{v} \times \mathbf{w}$.
    3. Pick any point P on either line ℓ_1 or ℓ_2.
    4. Using normal vector $\mathbf{n}$ and point P, follow CASE I.

- **EQUATION OF LINE PASSING THRU A POINT THAT'S ORTHOGONAL TO A PLANE:**

  Given point $P_0(x_0, y_0, z_0)$ and plane $\mathbb{P} : Ax + By + Cz + D = 0$

  1. Form normal vector to the plane: $\mathbf{n} = \langle A, B, C \rangle$.
  2. Equation of line is: $\begin{cases} x = x_0 + At \\ y = y_0 + Bt \\ z = z_0 + Ct \\ t \in \mathbb{R} \end{cases}$
  3. Substitute equation of line into equation of plane and solve for the parameter, t.
  4. Plug value of parameter t into equation of line to determine the point where the line intersects the plane.

- **DISTANCE BETWEEN A POINT AND A PLANE:**

  Given point P and plane $\mathbb{P} : Ax + By + Cz + D = 0$

  1. Form normal vector to the plane: $\mathbf{n} = \langle A, B, C \rangle$.
  2. Pick any point Q on the plane.
  3. Form vector $\mathbf{QP}$.
  4. $\Big(\text{Distance between point } P \text{ and plane } \mathbb{P}\Big) = \dfrac{|\mathbf{QP} \cdot \mathbf{n}|}{||\mathbf{n}||}$

- **DISTANCE BETWEEN TWO PARALLEL PLANES:**

  Given two parallel planes $\mathbb{P}_1 : A_1x + B_1y + C_1z + D_1 = 0$, $\mathbb{P}_2 : A_2x + B_2y + C_2z + D_2 = 0$

  1. Form normal vector to plane $\mathbb{P}_1$: $\mathbf{n} = \langle A_1, B_1, C_1 \rangle$
  2. Pick any point P on plane $\mathbb{P}_2$.
  3. Find distance between point P and plane $\mathbb{P}_1$.

EX 9.6.1: Find two normal vectors to the planes: (a) $4x - 3y - 6z + 20 = 0$ (b) $8y - z - 3 = 0$ (c) $x = 0$

EX 9.6.2: Find the equation for the plane normal to vector $\mathbf{n} = \langle 1, 2, 3 \rangle$ and containing point $P(-3, 4, -2)$.

EX 9.6.3: Find the equation for the plane normal to vector $\mathbf{n} = 7\widehat{\mathbf{j}} - 9\widehat{\mathbf{k}}$ and containing point $P(4, 4, -7)$.

EX 9.6.4: Find the equation for the plane normal to vector $\mathbf{n} = 8\widehat{\mathbf{i}}$ and containing point $P(-9, -11, -13)$.

EX 9.6.5: Find the equation for the plane containing points $P(1, 2, -2)$, $Q(-3, 1, 1)$, and $R(1, 2, 3)$.

EX 9.6.6: Find the equation of the plane determined by the intersecting lines $\ell_1 : \begin{cases} x = 7t \\ y = 1 - 8t \\ z = -1 - 5t \\ t \in \mathbb{R} \end{cases}$ and $\ell_2 : \begin{cases} x = 4 + 3s \\ y = -2 + s \\ z = 5 + 9s \\ s \in \mathbb{R} \end{cases}$

EX 9.6.7: (a) Find an equation of the line containing point $P(10, -3, -1)$ that's orthogonal to plane $x + 6y - 3z - 18 = 0$.

(b) Where does the line intersect the plane?

EX 9.6.8: Find the distance from the point $P(3, -9, 4)$ to the plane $3x - y + 5z - 11 = 0$.

EX 9.6.9: Find the distance between parallel planes $\mathbb{P}_1 : x - 7y + 8z - 2 = 0$ and $\mathbb{P}_2 : -3x + 21y - 24z + 11 = 0$.