SOLID ANALYTIC GEOMETRY: QUADRIC SURFACES [SST 9.7]

- **GENERAL FORM:** $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$, where $A, B, \ldots, I, J \in \mathbb{R}$
  - The general form is much too general! Henceforth, focus only on the canonical forms below.

QUADRIC SURFACE	CANONICAL FORM(S)	KEY PROPERTIES
Sphere	$x^2 + y^2 + z^2 = r^2$	Radius: r
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Axial Radii: a, b, c
Parabolic Cylinder	$y = ax^2$ OR $x = by^2$ $z = by^2$ OR $y = cz^2$ $z = ax^2$ OR $x = cz^2$	Axis of Generation: Axis of omitted variable
Elliptic Cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Axis of Generation: Axis of omitted variable
Hyperbolic Cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ OR $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ OR $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$ $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ OR $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$	Axis of Generation: Axis of omitted variable
Elliptic Paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $y = \frac{x^2}{a^2} + \frac{z^2}{c^2}$ $x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$	Axis of Revolution: Linear term
Elliptic Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$	Axis of Revolution: Negative Square term
Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Axis of Revolution: Negative Square term
Hyperboloid of Two Sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Axis of Revolution: Positive Square term
Hyperbolic Paraboloid	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ OR $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ $y = \frac{x^2}{a^2} - \frac{z^2}{c^2}$ OR $y = \frac{z^2}{c^2} - \frac{x^2}{a^2}$ $x = \frac{z^2}{c^2} - \frac{y^2}{b^2}$ OR $x = \frac{y^2}{b^2} - \frac{z^2}{c^2}$	

REMARK: All quadric surfaces tabulated above are centered at the origin $(0, 0, 0)$.

- **QUADRIC SURFACE IDENTIFICATION → HEURISTICS:**
  - Collect & isolate the squared-variable terms, then produce the **canonical form**.
  - **missing variable** $\overset{THINK}{\Longrightarrow}$ "**cylinder**".
  - **linear term** $\overset{THINK}{\Longrightarrow}$ "**parabolic**" or "**paraboloid**".
  - **all positive squared-variables** $\overset{THINK}{\Longrightarrow}$ "**elliptic**" or "**ellipsoid**".
  - **any negative squared-variables** $\overset{THINK}{\Longrightarrow}$ "**hyperbolic**" or "**hyperboloid**".

EX 9.7.1: Identify & characterize quadric surface: $x - 12z^2 = 0$

EX 9.7.2: Identify & characterize quadric surface: $x^2 + y^2 = 64 - z^2$

EX 9.7.3: Identify & characterize quadric surface: $45x^2 + 36y^2 + 20z^2 - 180 = 0$

EX 9.7.4: Identify & characterize quadric surface: $99z - 11x^2 - 9y^2 = 0$

EX 9.7.5: Identify & characterize quadric surface: $7y^2 - \frac{9}{16}x^2 - \frac{63}{16} = 0$

EX 9.7.6: Identify & characterize quadric surface: $3y^2 + 25z^2 - 75 = 0$

EX 9.7.7: Identify & characterize quadric surface: $\frac{9}{4}x^2 - 18y^2 + 8z^2 = 0$

EX 9.7.8: Identify & characterize quadric surface: $\pi\sqrt{5}x^2 + 4\sqrt{5}y^2 - 4\pi z^2 - 4\pi\sqrt{5} = 0$

EX 9.7.9: Identify & characterize quadric surface: $\pi^{8/5}x^2 - \pi^3 z^2 - \pi^{23/5} y = 0$

EX 9.7.10: Identify & characterize quadric surface: $\frac{\sqrt[3]{10}}{5}x^2 - \left(\sqrt[3]{10}\right) Qy^2 - \frac{Q}{5}z^2 - \frac{\sqrt[3]{10}}{5}Q = 0$, where $Q > 0$

$\boxed{\textbf{EX 9.7.11:}}$ Given the quadric surface $x = 2z^2$,

$\boxed{\text{(a)}}$ Find the intersection of the quadric surface with the plane $z = 3$

$\boxed{\text{(b)}}$ Find the intersection of the quadric surface with the plane $y = -1$

$\boxed{\text{(c)}}$ Find the intersection of the quadric surface with the plane $x = -2$

$\boxed{\text{(d)}}$ Find the intersection of the quadric surface with the plane $x = 0$

$\boxed{\text{(e)}}$ Find the intersection of the quadric surface with the plane $x = 2$

EX 9.7.12: Given the quadric surface $\dfrac{x^2}{2} - \dfrac{y^2}{3} - \dfrac{z^2}{4} = 1$,

(a) Find the intersection of the quadric surface with the plane $x = 0$

(b) Find the intersection of the quadric surface with the plane $x = \sqrt{2}$

(c) Find the intersection of the quadric surface with the plane $x = 4$

(d) Find the intersection of the quadric surface with the plane $z = 1$

(e) Find the intersection of the quadric surface with the plane $y = -\sqrt{3}$

EX 9.7.13: Given the quadric surface $\dfrac{x^2}{4} + \dfrac{y^2}{9} - \dfrac{z^2}{16} = 0$,

(a) Find the intersection of the quadric surface with the plane $z = 8$

(b) Find the intersection of the quadric surface with the plane $z = 0$

(c) Find the intersection of the quadric surface with the plane $y = -3$

(d) Find the intersection of the quadric surface with the plane $y = 0$

(e) Find the intersection of the quadric surface with the plane $x = 0$

(f) Find the intersection of the quadric surface with the plane $x = -2$