EX 11.8.3: Using one Lagrange Multiplier λ, minimize $f(x,y,z) = x - 2y + z$ subject to constraint $x^2 + 2y^2 + 3z^2 = 6$.

Let $g(x,y,z)$ be the LHS of the constraint: $g(x,y,z) = x^2 + 2y^2 + 3z^2$

Compute the gradients of f & g:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 1, -2, 1 \rangle \qquad \nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \langle 2x, 4y, 6z \rangle$$

Solve the nonlinear system:
$$\begin{cases} \nabla f &= \lambda \nabla g \\ g(x,y,z) &= 6 \end{cases} \implies \begin{cases} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x,y,z) &= 6 \end{cases} \implies \begin{cases} 1 &= 2x\lambda \\ -2 &= 4y\lambda \\ 1 &= 6z\lambda \\ x^2+2y^2+3z^2 &= 6 \end{cases}$$

Observe that $x \neq 0$, $y \neq 0$, $z \neq 0$, and $\lambda \neq 0$ (Do you see why??)

This observation allows us to safely divide by any of the variables, if necessary.

In the top three equations, solve for each variable in terms of λ:

$$1 = 2x\lambda \implies x = \frac{1}{2\lambda}, \qquad -2 = 4y\lambda \implies y = -\frac{1}{2\lambda}, \qquad 1 = 6z\lambda \implies z = \frac{1}{6\lambda}$$

Plug each λ-expression for x, y, z into the constraint (bottom equation):

$$\left(\frac{1}{2\lambda}\right)^2 + 2\left(-\frac{1}{2\lambda}\right)^2 + 3\left(\frac{1}{6\lambda}\right)^2 = 6 \implies \frac{1}{4}\cdot\frac{1}{\lambda^2} + \frac{1}{2}\cdot\frac{1}{\lambda^2} + \frac{1}{12}\cdot\frac{1}{\lambda^2} = 6 \implies \frac{5}{6}\cdot\frac{1}{\lambda^2} = 6 \implies \lambda^2 = \frac{5}{36} \implies \lambda = \pm\frac{\sqrt{5}}{6}$$

Find the corresponding values of x, y, z:

$$\lambda = -\frac{\sqrt{5}}{6} \implies x = \frac{1}{2\lambda} = \frac{1}{2}\left(-\frac{\sqrt{5}}{6}\right) = -\frac{3}{\sqrt{5}}, \quad y = -\frac{1}{2\lambda} = -\frac{1}{2}\left(-\frac{\sqrt{5}}{6}\right) = \frac{3}{\sqrt{5}}, \quad z = \frac{1}{6\lambda} = \frac{1}{6}\left(-\frac{\sqrt{5}}{6}\right) = -\frac{1}{\sqrt{5}}$$

$$\lambda = \frac{\sqrt{5}}{6} \implies x = \frac{1}{2\lambda} = \frac{1}{2}\left(\frac{\sqrt{5}}{6}\right) = \frac{3}{\sqrt{5}}, \quad y = -\frac{1}{2\lambda} = -\frac{1}{2}\left(\frac{\sqrt{5}}{6}\right) = -\frac{3}{\sqrt{5}}, \quad z = \frac{1}{6\lambda} = \frac{1}{6}\left(\frac{\sqrt{5}}{6}\right) = \frac{1}{\sqrt{5}}$$

Therefore, the CCP's of f are $\left(-\frac{3}{\sqrt{5}}, \frac{3}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$, $\left(\frac{3}{\sqrt{5}}, -\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

Finally, build a table computing f at each CCP:

(x,y,z)	$\left(-\frac{3}{\sqrt{5}}, \frac{3}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$	$\left(\frac{3}{\sqrt{5}}, -\frac{3}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
$f(x,y,z)$	$-\frac{10}{\sqrt{5}}$	$\frac{10}{\sqrt{5}}$
Result	**Abs Min**	

EX 11.8.4: Using one Lagrange Multiplier λ, maximize $f(x,y,z) = xyz$ subject to constraint $x+y+z=1$, where $\begin{matrix} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{matrix}$

Let $g(x,y,z)$ be the LHS of the constraint: $g(x,y,z) = x+y+z$

Compute the gradients of f & g:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle yz, xz, xy \rangle \qquad \nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \langle 1,1,1 \rangle$$

Solve the nonlinear system: $\left\{ \begin{array}{rcl} \nabla f & = & \lambda \nabla g \\ g(x,y,z) & = & 1 \end{array} \right. \implies \left\{ \begin{array}{rcl} f_x & = & \lambda g_x \\ f_y & = & \lambda g_y \\ f_z & = & \lambda g_z \\ g(x,y,z) & = & 1 \end{array} \right. \implies \left\{ \begin{array}{rcl} yz & = & \lambda \\ xz & = & \lambda \\ xy & = & \lambda \\ x+y+z & = & 1 \end{array} \right.$

Now, $\left\{ \begin{array}{rcl} yz & = & \lambda \\ xz & = & \lambda \\ xy & = & \lambda \\ x+y+z & = & 1 \end{array} \right. \implies yz = xz = xy \implies \begin{pmatrix} yz = xz \\ \textbf{and} \\ xz = xy \end{pmatrix} \implies \begin{pmatrix} yz - xz = 0 \\ \textbf{and} \\ xz - xy = 0 \end{pmatrix} \implies \begin{pmatrix} z(y-x) = 0 \\ \textbf{and} \\ x(z-y) = 0 \end{pmatrix}$

$$\implies \begin{pmatrix} z = 0 \textbf{ or } y - x = 0 \\ \textbf{and} \\ x = 0 \textbf{ or } z - y = 0 \end{pmatrix} \implies \begin{pmatrix} z = 0 \textbf{ or } y = x \\ \textbf{and} \\ x = 0 \textbf{ or } z = y \end{pmatrix} \implies \begin{pmatrix} z = 0 \textbf{ and } x = 0 \\ \textbf{or} \\ z = 0 \textbf{ and } z = y \\ \textbf{or} \\ y = x \textbf{ and } x = 0 \\ \textbf{or} \\ y = x \textbf{ and } z = y \end{pmatrix}$$

Apply each of these four cases to the constraint (bottom equation) and solve for x, y, z:

(CASE I) Suppose $z = 0$ and $x = 0$:

Then, $x+y+z=1 \implies (0)+y+(0)=1 \implies y=1 \implies (0,1,0)$ **is a CCP of** f

(CASE II) Suppose $z = 0$ and $z = y$:

Then, $y = 0$ **and** $x+y+z=1 \implies x+(0)+(0)=1 \implies x=1 \implies (1,0,0)$ **is a CCP of** f

(CASE III) Suppose $y = x$ and $x = 0$:

Then, $y = 0$ **and** $x+y+z=1 \implies (0)+(0)+z=1 \implies z=1 \implies (0,0,1)$ **is a CCP of** f

(CASE IV) Suppose $y = x$ and $z = y$:

Then, $x = y = z$ **and** $x+y+z=1 \implies x+x+x=1 \implies 3x=1 \implies x=\frac{1}{3} \implies y=\frac{1}{3}$ **and** $z=\frac{1}{3}$

$\implies \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ **is a CCP of** f

Therefore, the CCP's of f are $(1,0,0)$, $(0,1,0)$ $(0,0,1)$, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Finally, build a table computing f at each CCP:

(x,y,z)	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
$f(x,y,z)$	0	0	0	$\frac{1}{27}$
Result				**Abs Max**

EX 11.8.5: Find the largest product of positive real numbers x, y such that their sum is 60.

First of all, reframe this problem as a constrained optimization problem:

Using one Lagrange Multiplier λ, maximize $f(x,y) = xy$ subject to constraint $x + y = 60$, where $x \geq 0$, $y \geq 0$

Let $g(x,y)$ be the LHS of the constraint: $g(x,y) = x + y$

Compute the gradients of f & g:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle y, x \rangle \qquad \nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle = \langle 1, 1 \rangle$$

Solve the nonlinear system: $\begin{cases} \nabla f &= \lambda \nabla g \\ g(x,y) &= 60 \end{cases} \implies \begin{cases} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= 60 \end{cases} \implies \begin{cases} y &= \lambda \\ x &= \lambda \\ x + y &= 60 \end{cases}$

Plug the top two equations into the constraint (bottom equation) and solve for λ:

$x + y = 60 \implies (\lambda) + (\lambda) = 60 \implies 2\lambda = 60 \implies \lambda = 30$

Find the corresponding values of x, y:

$x = \lambda = 30$ **and** $y = \lambda = 30 \implies (30, 30)$ **is the only CCP of f**

Since there is only one CCP, pick another point on the constraint:

Remember that $x \geq 0$, $y \geq 0$ (see the "reframing problem as constrained optimization" above)

Let $x = 20$, then $(20) + y = 60 \implies y = 40$

Therefore, $(20, 40)$ is a point on the constraint

Finally, build a table computing f at each CCP:

(x,y)	$(\mathbf{30}, \mathbf{30})$	$(20, 40)$
$f(x,y)$	**900**	800
Result	**Abs Max**	

EX 11.8.6: Find the smallest sum of positive real numbers x, y, z such that their product is 64.

First of all, reframe this problem as a constrained optimization problem:

Using one Lagrange Multiplier λ, minimize $f(x, y) = x + y + z$ s.t. constraint $xyz = 64$, where $x \geq 0, y \geq 0, z \geq 0$

Let $g(x, y, z)$ be the LHS of the constraint: $g(x, y, z) = xyz$

Compute the gradients of f & g:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 1, 1, 1 \rangle \qquad \nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \langle yz, xz, xy \rangle$$

Solve the nonlinear system:

$$\begin{cases} \nabla f &= \lambda \nabla g \\ g(x,y,z) &= 64 \end{cases} \implies \begin{cases} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x,y,z) &= 64 \end{cases} \implies \begin{cases} 1 &= yz\lambda \\ 1 &= xz\lambda \\ 1 &= xy\lambda \\ xyz &= 64 \end{cases}$$

Observe that $x \neq 0$, $y \neq 0$, $z \neq 0$, $\lambda \neq 0$ (Do you see why??)

This observation allows us to safely divide by any of the variables, if necessary.

Divide both sides of the top three equations by λ:

$$\begin{cases} 1 = yz\lambda \\ 1 = xz\lambda \\ 1 = xy\lambda \end{cases} \implies \begin{cases} 1/\lambda = yz \\ 1/\lambda = xz \\ 1/\lambda = xy \end{cases}$$

Now, since the LHS's are all equal, equate the RHS's to each other:

$yz = xz = xy$

Now, unpack this "chain of equalities" into a pair of equations & then simplify:

$$yz = xz = xy \implies \begin{pmatrix} yz = xz \\ \textbf{and} \\ xz = xy \end{pmatrix} \implies \begin{pmatrix} yz - xz = 0 \\ \textbf{and} \\ xz - xy = 0 \end{pmatrix} \implies \begin{pmatrix} (y-x)z = 0 \\ \textbf{and} \\ x(z-y) = 0 \end{pmatrix} \implies \begin{pmatrix} y = x \ \textbf{or}\ \cancel{z=0} \\ \textbf{and} \\ \cancel{x=0}\ \textbf{or}\ z = y \end{pmatrix}$$

$$\implies \begin{pmatrix} y = x \\ \textbf{and} \\ z = y \end{pmatrix} \implies x = y = z$$

Find the corresponding values of x, y, z by using the constraint (bottom equation):

$xyz = 64 \implies x(x)(x) = 64 \implies x^3 = 64 \implies x = 4 \implies y = 4$ **and** $z = 4$

Therefore, the only CCP of f is $(4, 4, 4)$.

Since there is only one CCP, pick another point on the constraint:

Remember that $x > 0$, $y > 0$, $z > 0$ (see the "reframing as constrained opt." & observation above)

Let $x = 1$ and $y = 1$, then $(1)(1)z = 64 \implies z = 64$

Therefore, $(1, 1, 64)$ is a point on the constraint

Finally, build a table computing f at each CCP:

(x,y,z)	$\mathbf{(4,4,4)}$	$(1,1,64)$
$f(x,y,z)$	**12**	66
Result	**Abs Min**	