Vector Functions: Algebra & Limits

Calculus III

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TTU

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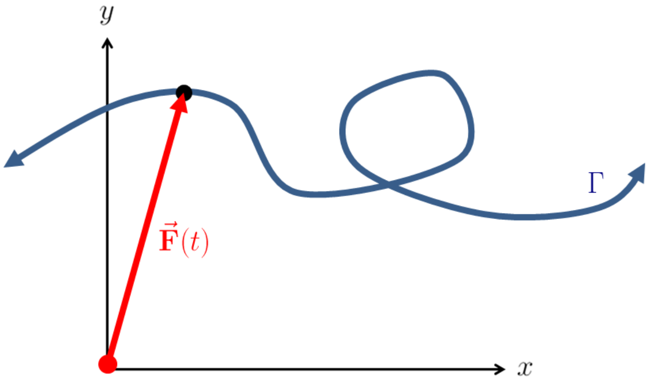
2D Vector Functions (Definition)

A **2D vector function** is a 2D parametric curve ”wrapped” in a vector.



- $\vec{\mathbf{F}}(t) := \langle f_1(t), f_2(t) \rangle = f_1(t)\widehat{\mathbf{i}} + f_2(t)\widehat{\mathbf{j}}$
- $\text{Dom}(\vec{\mathbf{F}}) := \text{Dom}(f_1) \cap \text{Dom}(f_2)$

2D Vector Functions (Definition)

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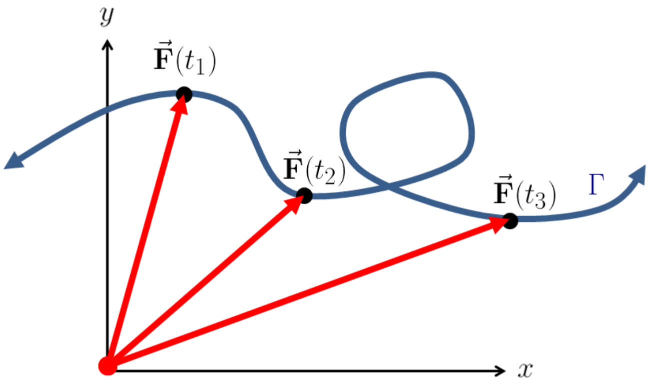
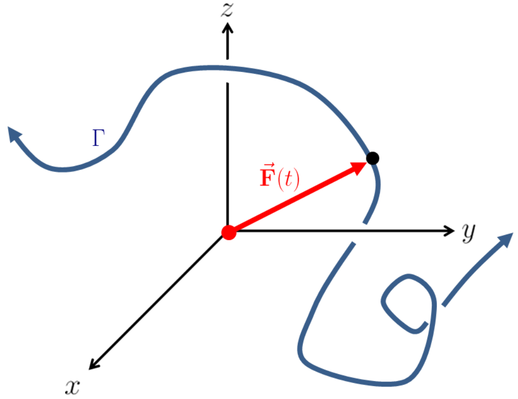


- Here, $t_1 < t_2 < t_3$.
- $\vec{\mathbf{F}}(t)$ traces out a **curve** (**trajectory**) on the xy-plane, labeled Γ ("Gamma")

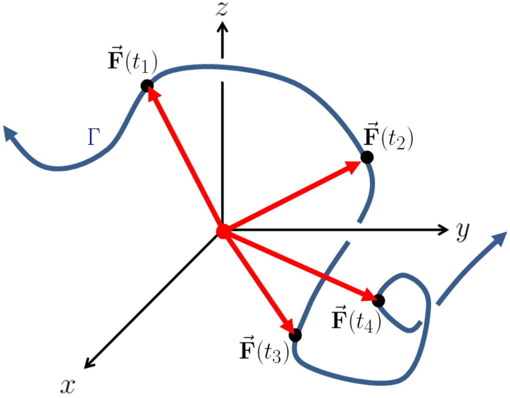
3D Vector Functions (Definition)

A **3D vector function** is a 3D parametric curve ”wrapped” in a vector.



- $\vec{\mathbf{F}}(t) := \langle f_1(t), f_2(t), f_3(t) \rangle = f_1(t)\widehat{\mathbf{i}} + f_2(t)\widehat{\mathbf{j}} + f_3(t)\widehat{\mathbf{k}}$
- $\text{Dom}(\vec{\mathbf{F}}) := \text{Dom}(f_1) \cap \text{Dom}(f_2) \cap \text{Dom}(f_3)$

A **3D vector function** is a 3D parametric curve "wrapped" in a vector.



- Here, $t_1 < t_2 < t_3 < t_4$.
- $\vec{\mathbf{F}}(t)$ traces out a **curve** (**trajectory**) in space, labeled Γ ("Gamma")

Vector Functions & Parametric Curves

Parametric curves can be represented very compactly with vector functions:

In $\mathbb{R}^2$: $\begin{cases} x = f(t) \\ y = g(t) \\ t \in I \end{cases} \iff \vec{\mathbf{R}}(t) = \langle f(t), g(t) \rangle \, , \; t \in I$

<u>REMARK:</u> If '$t \in I$' is omitted, then $t \in$ Dom$(f) \cap$ Dom(g)

In $\mathbb{R}^3$: $\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \\ t \in I \end{cases} \iff \vec{\mathbf{R}}(t) = \langle f(t), g(t), h(t) \rangle \, , \; t \in I$

<u>REMARK:</u> If '$t \in I$' is omitted, then $t \in$ Dom$(f) \cap$ Dom$(g) \cap$ Dom(h)

<u>NOTATION:</u> $\iff \equiv$ "is (logically) equivalent to" $\equiv$ "if and only if"

Vector Functions & Lines

In particular, lines can be represented very compactly with vector functions:

In $\mathbb{R}^2$: $\ell : \begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ t \in \mathbb{R} \end{cases} \iff \vec{\mathbf{L}}(t) = \langle x_0 + v_1 t, y_0 + v_2 t \rangle$

In $\mathbb{R}^3$: $\ell : \begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ z = z_0 + v_3 t \\ t \in \mathbb{R} \end{cases} \iff \vec{\mathbf{L}}(t) = \langle x_0 + v_1 t, y_0 + v_2 t, z_0 + v_3 t \rangle$

<u>NOTATION:</u> $\iff \equiv$ "is (logically) equivalent to" $\equiv$ "if and only if"

CONIC SECTION	RECT. FORM	PARAMETRIC FORM
Parabola	$y = Ax^2 + C$	$\vec{\mathbf{R}}(t) = \left\langle t, At^2 + C \right\rangle, \ t \in \mathbb{R}$
Parabola	$x = By^2 + D$	$\vec{\mathbf{R}}(t) = \left\langle Bt^2 + D, t \right\rangle, \ t \in \mathbb{R}$
Circle	$x^2 + y^2 = r^2$	$\vec{\mathbf{R}}(t) = \langle r\cos t, r\sin t \rangle, \ t \in [0, 2\pi]$
Ellipse	$\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$	$\vec{\mathbf{R}}(t) = \langle A\cos t, B\sin t \rangle, \ t \in [0, 2\pi]$
Hyperbola	$\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$	$\vec{\mathbf{R}}(t) = \langle A\sec t, B\tan t \rangle$, Left: $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ Right: $t \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$
Hyperbola	$\dfrac{y^2}{B^2} - \dfrac{x^2}{A^2} = 1$	$\vec{\mathbf{R}}(t) = \langle A\tan t, B\sec t \rangle$, Btm: $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ Top: $t \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$

REMARK: The **restricted** t-values ensure the curve is traced **only once**.

REMARK: Parameterizations are not unique (but, the above are convention.)

Vector Functions (vs Scalar Functions)

FUNCTION TYPE	PROTOTYPE	MEANING
Scalar Function	$y = f(x)$	f maps scalar $\to$ scalar
Scalar Function	$x = g(y)$	g maps scalar $\to$ scalar
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F}$ maps scalar $\to$ 2D vector
3D Vector Function	$\mathbf{H}(t) = \langle h_1(t), h_2(t), h_3(t) \rangle$	$\mathbf{H}$ maps scalar $\to$ 3D vector

Vector Functions (vs Scalar Functions)

FUNCTION TYPE	PROTOTYPE	MAPPING
Scalar Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$
Scalar Function	$x = g(y)$	$g : \mathbb{R} \to \mathbb{R}$
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$
3D Vector Function	$\mathbf{H}(t) = \langle h_1(t), h_2(t), h_3(t) \rangle$	$\mathbf{H} : \mathbb{R} \to \mathbb{R}^3$

Vector Functions (Algebra)

Let $\mathbf{F}(t), \mathbf{G}(t), \mathbf{H}(t)$ be vector functions & f, h be scalar functions. Then:

OPERATION	FORMULA	REMARKS
Addition/Subtraction	$\mathbf{H}(t) = \mathbf{F}(t) \pm \mathbf{G}(t)$	$\text{Dom}(\mathbf{H}) = \text{Dom}(\mathbf{F}) \cap \text{Dom}(\mathbf{G})$
Scalar Multiplication	$\mathbf{H}(t) = f(t)\mathbf{F}(t)$	$\text{Dom}(\mathbf{H}) = \text{Dom}(f) \cap \text{Dom}(\mathbf{F})$
Dot Product	$h(t) = \mathbf{F}(t) \cdot \mathbf{G}(t)$	$\text{Dom}(h) = \text{Dom}(\mathbf{F}) \cap \text{Dom}(\mathbf{G})$
Cross Product*	$\mathbf{H}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$	$\text{Dom}(\mathbf{H}) = \text{Dom}(\mathbf{F}) \cap \text{Dom}(\mathbf{G})$

* Cross products are defined only for **3D** vector functions!

i.e., operations on vector functions behave exactly like operations on vectors.

Vector Functions (Limits)

Definition

Let vector function $\mathbf{F}(t) := \langle f_1(t), f_2(t), f_3(t) \rangle$.
Moreover, suppose the scalar limits $\lim_{t\to t_0} f_1(t)$, $\lim_{t\to t_0} f_2(t)$, $\lim_{t\to t_0} f_3(t)$ are all **finite**.

Then:

$$\lim_{t\to t_0} \mathbf{F}(t) := \left\langle \lim_{t\to t_0} f_1(t), \lim_{t\to t_0} f_2(t), \lim_{t\to t_0} f_3(t) \right\rangle$$

Otherwise, if any of the above three scalar limits are $-\infty, +\infty,$ or DNE*, then $\lim_{t\to t_0} \mathbf{F}(t) =$ DNE

REMARK: The limit of a vector function, if it exists, is a **vector**.

* DNE $\equiv$ **D**oes **N**ot **E**xist

Vector Functions (Limits)

Let $\mathbf{F}(t), \mathbf{G}(t)$ be vector functions & f be a scalar function. Then:

LIMIT RULE	**FORMULA**
Sum/Diff Rule	$\lim_{t\to t_0} [\mathbf{F}(t) \pm \mathbf{G}(t)] = \left[\lim_{t\to t_0} \mathbf{F}(t)\right] \pm \left[\lim_{t\to t_0} \mathbf{G}(t)\right]$
Scalar Multiple Rule	$\lim_{t\to t_0} [f(t)\mathbf{F}(t)] = \left[\lim_{t\to t_0} f(t)\right] \left[\lim_{t\to t_0} \mathbf{F}(t)\right]$
Dot Product Rule	$\lim_{t\to t_0} [\mathbf{F}(t) \cdot \mathbf{G}(t)] = \left[\lim_{t\to t_0} \mathbf{F}(t)\right] \cdot \left[\lim_{t\to t_0} \mathbf{G}(t)\right]$
Cross Product Rule*	$\lim_{t\to t_0} [\mathbf{F}(t) \times \mathbf{G}(t)] = \left[\lim_{t\to t_0} \mathbf{F}(t)\right] \times \left[\lim_{t\to t_0} \mathbf{G}(t)\right]$

* Cross products are defined only for **3D** vector functions!

Vector Functions (Continuity)

Definition

Let $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$. Then:
$\mathbf{F}(t)$ is **continous at a point** t_0, denoted $\mathbf{F} \in C(\{t_0\})$, if

$$t_0 \in \text{Dom}(\mathbf{F}) \text{ AND } \lim_{t \to t_0} \mathbf{F}(t) = \mathbf{F}(t_0)$$

$\mathbf{F}(t)$ is **continous at a set** $S \subseteq \mathbb{R}$, denoted $\mathbf{F} \in C(S)$, if

$$\text{its componenents } f_1, f_2, f_3 \in C(S)$$

To determine the set of continuity for vector function $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$:

- Find the set of continuity for each component of $\mathbf{F}(t)$:
  $f_1 \in C(S_1),\ f_2 \in C(S_2),\ f_3 \in C(S_3)$
- Take the **intersection** every component's set of continuity:
  $\mathbf{F} \in C(S_1 \cap S_2 \cap S_3)$

Fin.