Vector Functions: Calculus & Kinematics

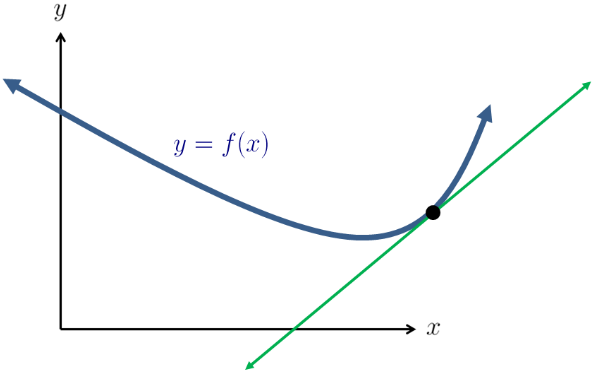
Calculus III

Josh Engwer

TTU

15 September 2014

Scalar Functions (1^{st} Derivative Definition)



Recall from Calculus I: $$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

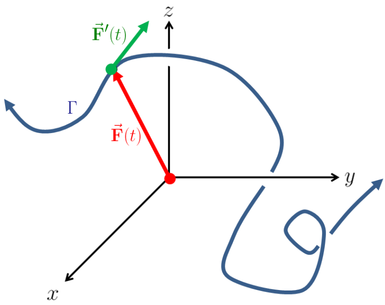
Geometrically, $f'(x)$ represents the **slope of the tangent line** to curve $y = f(x)$

Vector Functions (1^{st} Derivative Derivation)

Given vector function $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$,

$$\begin{aligned}
\mathbf{F}'(t) &:= \lim_{\Delta t \to 0} \frac{\mathbf{F}(t+\Delta t) - \mathbf{F}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}, \frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}, \frac{f_3(t+\Delta t) - f_3(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f_1(t+\Delta t) - f_1(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{f_2(t+\Delta t) - f_2(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{f_3(t+\Delta t) - f_3(t)}{\Delta t} \right\rangle \\
&:= \langle f_1'(t), f_2'(t), f_3'(t) \rangle
\end{aligned}$$

Vector Functions (1^{st} Derivative Definition)



Definition

Let vector function $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$ trace curve Γ. Then:

$$\mathbf{F}'(t) := \langle f_1'(t), f_2'(t), f_3'(t) \rangle$$

Geometrically, $\mathbf{F}'(t)$ represents the **tangent vector** to the curve Γ.

Vector Functions (Derivative Rules)

Let $\mathbf{F}(t), \mathbf{G}(t)$ be vector functions & h be a scalar function. Then:

DERIVATIVE RULE	FORMULA
Sum/Diff Rule	$\frac{d}{dt}\left[\mathbf{F}(t) \pm \mathbf{G}(t)\right] = \mathbf{F}'(t) \pm \mathbf{G}'(t)$
Scalar Multiple Rule	$\frac{d}{dt}\left[h(t)\mathbf{F}(t)\right] = h'(t)\mathbf{F}(t) + h(t)\mathbf{F}'(t)$
Dot Product Rule	$\frac{d}{dt}\left[\mathbf{F}(t) \cdot \mathbf{G}(t)\right] = \mathbf{F}'(t) \cdot \mathbf{G}(t) + \mathbf{F}(t) \cdot \mathbf{G}'(t)$
Cross Product Rule*	$\frac{d}{dt}\left[\mathbf{F}(t) \times \mathbf{G}(t)\right] = \mathbf{F}'(t) \times \mathbf{G}(t) + \mathbf{F}(t) \times \mathbf{G}'(t)$
Chain Rule	$\frac{d}{dt}\left[\mathbf{F}\left[h(t)\right]\right] = \mathbf{F}'\left[h(t)\right] h'(t)$

* Cross products are defined only for **3D** vector functions!

Vector Functions (Higher-Order Derivatives)

ORDER	DEFINITION	LEIBNIZ NOTATION
1^{st} Derivative	$\mathbf{F}'(t) := \langle f_1'(t), f_2'(t), f_3'(t)\rangle$	$\frac{d\mathbf{F}}{dt}$
2^{nd} Derivative	$\mathbf{F}''(t) := \langle f_1''(t), f_2''(t), f_3''(t)\rangle$	$\frac{d^2\mathbf{F}}{dt^2}$
3^{rd} Derivative	$\mathbf{F}'''(t) := \langle f_1'''(t), f_2'''(t), f_3'''(t)\rangle$	$\frac{d^3\mathbf{F}}{dt^3}$
4^{th} Derivative	$\mathbf{F}^{(4)}(t) := \left\langle f_1^{(4)}(t), f_2^{(4)}(t), f_3^{(4)}(t)\right\rangle$	$\frac{d^4\mathbf{F}}{dt^4}$
$\vdots$	$\vdots$	$\vdots$
n^{th} Derivative	$\mathbf{F}^{(n)}(t) := \left\langle f_1^{(n)}(t), f_2^{(n)}(t), f_3^{(n)}(t)\right\rangle$	$\frac{d^n\mathbf{F}}{dt^n}$

Notation for Continuous Derivatives (Calculus I)

Recall the notation for a **continuous function** f on a set S: $f \in C(S)$

Definition

Given function $f(x)$ and set $S \subseteq \mathbb{R}$. Then:

$$f \in C^1(S) \iff f, f' \in C(S)$$

$$f \in C^2(S) \iff f, f', f'' \in C(S)$$

$$f \in C^3(S) \iff f, f', f'', f''' \in C(S)$$

$$f \in C^4(S) \iff f, f', f'', f''', f^{(4)} \in C(S)$$

$$f \in C^k(S) \iff f, f', f'', \cdots, f^{(k-1)}, f^{(k)} \in C(S)$$

Vector Functions (Differentiability)

Definition

Let vector function $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$. Then:
$\mathbf{F}(t)$ is **differentiable** on a set $S \subseteq \mathbb{R}$, denoted $\mathbf{F} \in C^1(S)$, if

$$\text{components } f_1, f_2, f_3 \in C^1(S)$$

$\mathbf{F}(t)$ is **twice differentiable** on a set $S \subseteq \mathbb{R}$, denoted $\mathbf{F} \in C^2(S)$, if

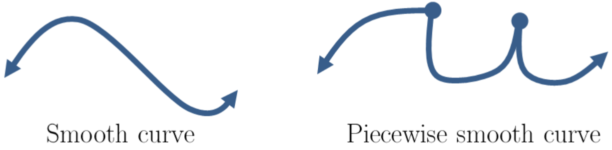
$$\text{components } f_1, f_2, f_3 \in C^2(S)$$

$\mathbf{F}(t)$ is **three times differentiable** on a set $S \subseteq \mathbb{R}$, denoted $\mathbf{F} \in C^3(S)$, if

$$\text{components } f_1, f_2, f_3 \in C^3(S)$$

$\mathbf{F}(t)$ is k-**times differentiable** on a set $S \subseteq \mathbb{R}$, denoted $\mathbf{F} \in C^k(S)$, if

$$\text{components } f_1, f_2, f_3 \in C^k(S)$$



Definition

Let vector function $\mathbf{F}(t)$ trace curve Γ. Then:
$\mathbf{F}(t)$ (curve Γ) is **smooth on interval** (a,b) if

$$\left[\mathbf{F} \in C^1(a,b) \text{ AND } \mathbf{F}'(t) \neq \vec{\mathbf{0}} \quad \forall t \in (a,b)\right]$$

$\mathbf{F}(t)$ (curve Γ) is **piecewise smooth on** (a,b) if

$\mathbf{F}(t)$ is smooth on (a,b) except at a finite # of points.

REMARK: A smooth curve has no **corners** or **cusps**. (see top-right curve)

Vector Functions (Integration)

Definition

Let vector function $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$. Then:

$$\int_a^b \mathbf{F}(t)\, dt := \left\langle \int_a^b f_1(t)\, dt, \int_a^b f_2(t)\, dt, \int_a^b f_3(t)\, dt \right\rangle$$

$$\int \mathbf{F}(t)\, dt := \left\langle \int f_1(t)\, dt, \int f_2(t)\, dt, \int f_3(t)\, dt \right\rangle + \vec{\mathbf{C}}$$

where $\vec{\mathbf{C}} := \langle C_1, C_2, C_3 \rangle$ is the **constant vector of integration.**

Vector Functions (3D Kinematics)

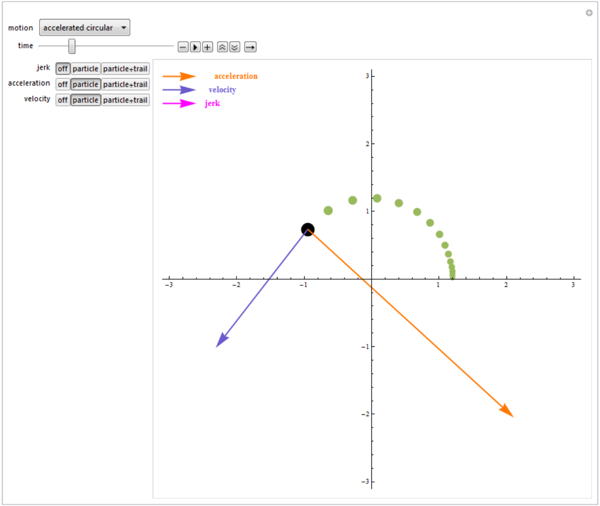
Definition

SETUP: Given a **particle** in 3D with **position vector** $\mathbf{R}(t) = \langle R_1(t), R_2(t), R_3(t) \rangle$

Trajectory			$\Gamma :=$ Graph of position vector $\mathbf{R}(t)$		
Velocity	**Speed**	**Direction**	$\mathbf{V}(t) := \mathbf{R}'(t)$	$\|\|\mathbf{V}(t)\|\|$	$\dfrac{\mathbf{V}(t)}{\|\|\mathbf{V}(t)\|\|}$
Acceleration			$\mathbf{A}(t) := \mathbf{V}'(t) = \mathbf{R}''(t)$		

The particle is **stationary** $\iff \mathbf{V}(t) = \vec{\mathbf{0}}$

(DEMO) 3D KINEMATICS (click below):



Fin.