Vector Functions: **TNB**-Frame & Curvature

Calculus III

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TTU

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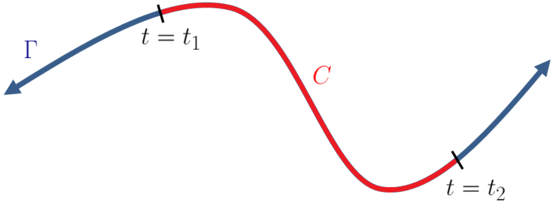
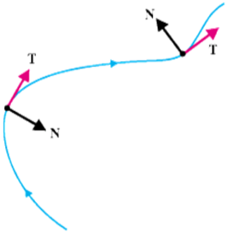
Vector Functions (Arc Length)

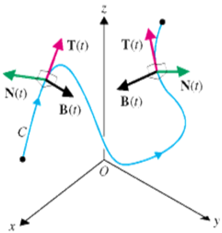


Definition

Let vector function $\mathbf{R}(t) = \langle x(t), y(t), z(t) \rangle$ trace a **piecewise smooth curve** Γ. Let C be the portion of curve Γ that's traced once over the interval $t \in [t_1, t_2]$. Then, the **arc length** of C is defined by:

$$\text{ArcLength}(C) := \int_{t_1}^{t_2} ||\mathbf{R}'(t)||\, dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$





Definition

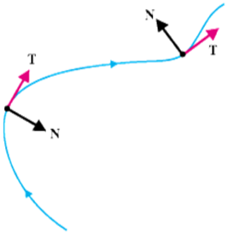
Let vector function $\mathbf{R}(t)$ trace a **smooth curve.** Then:

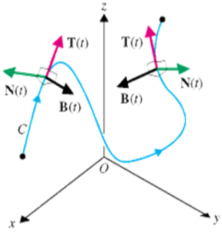
Unit Tangent Vector $\widehat{\mathbf{T}}(t) := \dfrac{\mathbf{R}'(t)}{||\mathbf{R}'(t)||}$ **Unit Normal Vector** $\widehat{\mathbf{N}}(t) := \dfrac{\widehat{\mathbf{T}}'(t)}{||\widehat{\mathbf{T}}'(t)||}$

Unit Binormal Vector $\widehat{\mathbf{B}}(t) := \widehat{\mathbf{T}}(t) \times \widehat{\mathbf{N}}(t)$

Collectively, these three vectors form the **TNB-Frame.**

TNB-Frame (Geometric Interpretation)





- The **TNB**-Frame is a **frame of reference**.
- $\widehat{\mathbf{T}}(t)$ points in the direction of motion along the curve.
- $\widehat{\mathbf{N}}(t)$ points in the direction the curve is bending.
- $\widehat{\mathbf{B}}(t)$ is **orthogonal** to both $\widehat{\mathbf{T}}(t)$ & $\widehat{\mathbf{N}}(t)$.
- $\widehat{\mathbf{N}}(t)$ is **orthogonal** to $\widehat{\mathbf{T}}(t)$.

The claim that $\widehat{\mathbf{N}}(t) \perp \widehat{\mathbf{T}}(t)$ is not obvious, so let's prove it....

TNB-Frame (Proof that $\widehat{\mathbf{N}}(t) \perp \widehat{\mathbf{T}}(t)$)

First, observe that $\widehat{\mathbf{T}}(t)$ has norm one.

$$\implies ||\widehat{\mathbf{T}}(t)|| = 1$$

$$\implies ||\widehat{\mathbf{T}}(t)||^2 = 1$$

$$\implies \widehat{\mathbf{T}}(t) \cdot \widehat{\mathbf{T}}(t) = 1 \qquad \left(\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2\right)$$

$$\implies \frac{d}{dt}\left[\widehat{\mathbf{T}}(t) \cdot \widehat{\mathbf{T}}(t)\right] = \frac{d}{dt}[1]$$

$$\implies \widehat{\mathbf{T}}'(t) \cdot \widehat{\mathbf{T}}(t) + \widehat{\mathbf{T}}(t) \cdot \widehat{\mathbf{T}}'(t) = 0$$

$$\implies \widehat{\mathbf{T}}'(t) \cdot \widehat{\mathbf{T}}(t) + \widehat{\mathbf{T}}'(t) \cdot \widehat{\mathbf{T}}(t) = 0 \qquad \left(\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}\right)$$

$$\implies 2\widehat{\mathbf{T}}'(t) \cdot \widehat{\mathbf{T}}(t) = 0$$

$$\implies \widehat{\mathbf{T}}'(t) \cdot \widehat{\mathbf{T}}(t) = 0$$

$$\implies \frac{\widehat{\mathbf{T}}'(t) \cdot \widehat{\mathbf{T}}(t)}{||\widehat{\mathbf{T}}'(t)||} = \frac{0}{||\widehat{\mathbf{T}}'(t)||}$$

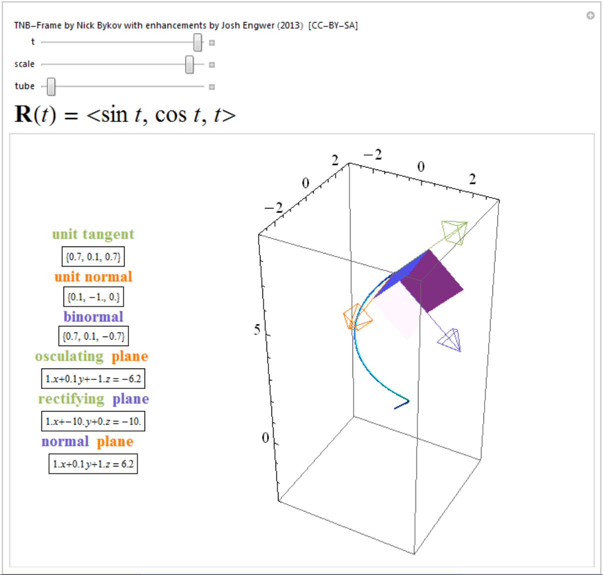
$$\implies \frac{\widehat{\mathbf{T}}'(t)}{||\widehat{\mathbf{T}}'(t)||} \cdot \widehat{\mathbf{T}}(t) = 0 \qquad \left(c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w}\right)$$

$$\implies \widehat{\mathbf{N}}(t) \cdot \widehat{\mathbf{T}}(t) = 0$$

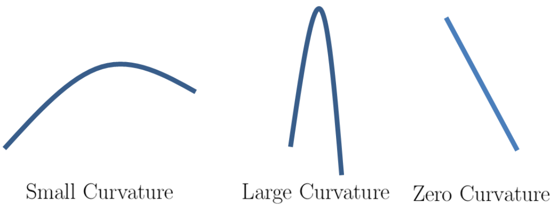
$$\implies \widehat{\mathbf{N}}(t) \perp \widehat{\mathbf{T}}(t)$$

QED

(DEMO) TNB-Frame (click below):



Curvature (Definition)



Definition

Let vector function $\mathbf{R}(t)$ trace a **smooth curve** Γ.
Then, the **curvature** of Γ is defined by:

$$\kappa(t) := \frac{||\widehat{\mathbf{T}}'(t)||}{||\mathbf{R}'(t)||}$$

where $\widehat{\mathbf{T}}(t)$ is the **unit tangent vector**.

Geometrically, curvature measures the curve's tendency to "bend."

"Curvature's Curse"

CURVE TYPE	PROTOTYPE	CURVATURE FORMULA
3D Vector Function	$\mathbf{R}(t) = \langle x(t), y(t), z(t) \rangle$	$\dfrac{\lvert\lvert \mathbf{T}'(t) \rvert\rvert}{\lvert\lvert \mathbf{R}'(t) \rvert\rvert}$ OR $\dfrac{\lvert\lvert \mathbf{R}'(t) \times \mathbf{R}''(t) \rvert\rvert}{\lvert\lvert \mathbf{R}'(t) \rvert\rvert^3}$
Rectangular Curve	$y = f(x)$	$\dfrac{\lvert f''(x) \rvert}{\left(1 + [f'(x)]^2\right)^{3/2}}$
Rectangular Curve	$x = g(y)$	$\dfrac{\lvert g''(y) \rvert}{\left(1 + [g'(y)]^2\right)^{3/2}}$
2D Parametric Curve	$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$	$\dfrac{\lvert x'y'' - y'x'' \rvert}{[(x')^2 + (y')^2]^{3/2}}$
Polar Curve	$r = f(\theta)$	$\dfrac{\left\lvert r^2 + 2(r')^2 - rr'' \right\rvert}{[r^2 + (r')^2]^{3/2}}$

A Cure for "Curvature's Curse"

CURVE TYPE	PROTOTYPE	CURVATURE FORMULA
3D Vector Function	$\mathbf{R}(t) = \langle x(t), y(t), z(t) \rangle$	$\frac{\|\|\mathbf{T}'(t)\|\|}{\|\|\mathbf{R}'(t)\|\|}$ or $\frac{\|\|\mathbf{R}'(t) \times \mathbf{R}''(t)\|\|}{\|\|\mathbf{R}'(t)\|\|^3}$
Rectangular Curve	$y = f(x)$	$\frac{\|f''(x)\|}{\left(1+[f'(x)]^2\right)^{3/2}}$
Rectangular Curve	$x = g(y)$	$\frac{\|g''(y)\|}{\left(1+[g'(y)]^2\right)^{3/2}}$
2D Parametric Curve	$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$	$\frac{\|x'y'' - y'x''\|}{[(x')^2 + (y')^2]^{3/2}}$
Polar Curve	$r = f(\theta)$	$\frac{\|r^2 + 2(r')^2 - rr''\|}{[r^2 + (r')^2]^{3/2}}$

The Cure: Throw away all these complex formulas for curvature.

A Cure for "Curvature's Curse"

CURVE TYPE	PROTOTYPE	VECTOR ENCAPSULATION
Rectangular Curve	$y = f(x)$	$\mathbf{R}(t) := \langle t, f(t), 0 \rangle$
Rectangular Curve	$x = g(y)$	$\mathbf{R}(t) := \langle g(t), t, 0 \rangle$
2D Parametric Curve	$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$	$\mathbf{R}(t) := \langle f(t), g(t), 0 \rangle$
Polar Curve	$r = f(\theta)$	$\mathbf{R}(t) := \langle f(t)\cos t, f(t)\sin t, 0 \rangle$

The Cure: Convert curve to 3D Vector Function by **vector encapsulation**.

Then use the simplest curvature formula: $\kappa(t) = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{R}'(t)||}$

Torsion (Definition)

Definition

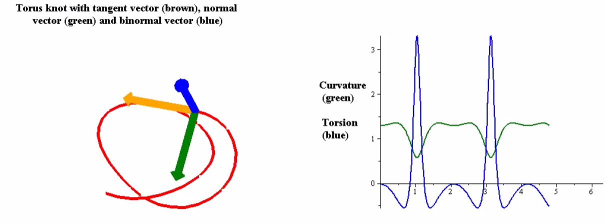
Let vector function $\mathbf{R}(t)$ trace a **smooth curve** Γ.
Then, the **torsion** of Γ is defined by:

$$\tau(t) := -\widehat{\mathbf{N}}(t) \cdot \widehat{\mathbf{B}}'(t)$$

where $\widehat{\mathbf{N}}(t)$ is the **unit normal vector** & $\widehat{\mathbf{B}}(t)$ is the **unit binormal vector**.

Geometrically, torsion measures the curve's tendency to "twist."

(DEMO) CURVATURE & TORSION (click below):



Fin.