Functions of Several Variables: Introduction

Calculus III

Josh Engwer

TTU

22 September 2014

Functions of Several Variables (Definitions)

Definition

A **function of 2 variables** $f(x,y)$ depends on **2 independent variables** x,y.

REMARK: There's no dependence between x and y.

EXAMPLE:
$T(x,t)$: Temperature of a thin rod (T) depends on position (x) and time (t).

Definition

A **function of 3 variables** $f(x,y,z)$ depends on **3 independent var's** x,y,z.

REMARK: There's no dependence among x, y and z.

EXAMPLE:
$T(x,y,t)$: Temperature of a plate (T) depends on position (x,y) and time (t).

"The Function Landscape"

FUNCTION TYPE	PROTOTYPE	MAPPING
(Scalar) Function	$y = f(x)$	f maps scalar $\to$ scalar
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F}$ maps scalar $\to$ 2D vector
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F}$ maps scalar $\to$ 3D vector
Function of 2 Variables	$z = f(x, y)$	f maps 2D point $\to$ scalar
Function of 3 Variables	$w = f(x, y, z)$	f maps 3D point $\to$ scalar

"The Function Landscape"

FUNCTION TYPE	PROTOTYPE	MAPPING
(Scalar) Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$
Function of 2 Variables	$z = f(x, y)$	$f : \mathbb{R}^2 \to \mathbb{R}$
Function of 3 Variables	$w = f(x, y, z)$	$f : \mathbb{R}^3 \to \mathbb{R}$

$\mathbb{R} :=$ The set of all scalars on the real line

$\mathbb{R}^2 :=$ The set of all ordered pairs (x, y) on the xy-plane (say "R Two")

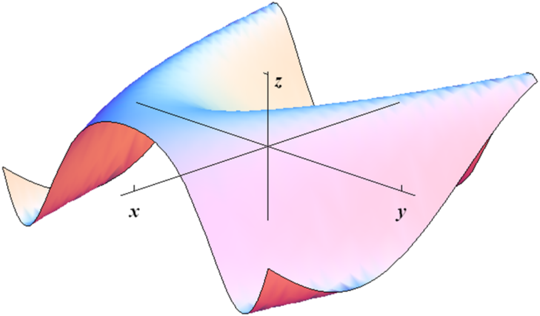
$\mathbb{R}^2 :=$ The set of all 2D vectors $\langle v_1, v_2 \rangle$ on the xy-plane

$\mathbb{R}^3 :=$ The set of all ordered triples (x, y, z) in xyz-space (say "R Three")

$\mathbb{R}^3 :=$ The set of all 3D vectors $\langle v_1, v_2, v_3 \rangle$ in xyz-space

Functions of Two Variables (Graphs)

$$f(x,y) = \sin\left(x + y^2\right)$$



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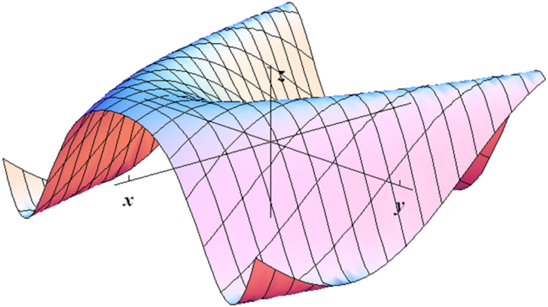

Functions of Two Variables (Graphs)

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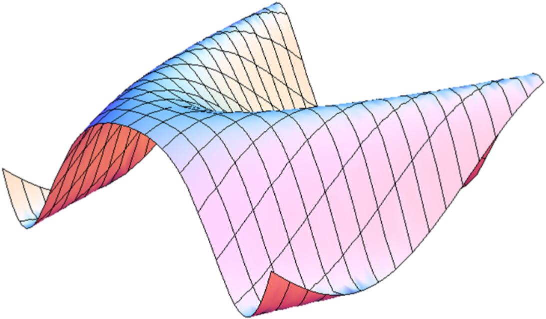
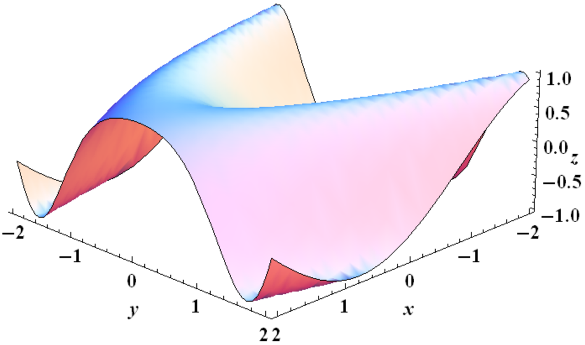
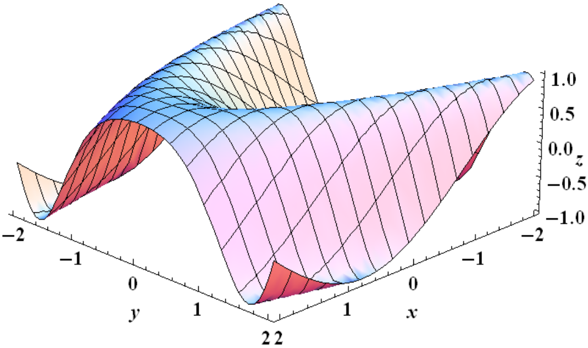


Functions of Two Variables (Graphs)



Functions of Two Variables (Graphs)



The graph of $w = f(x, y, z)$ is a **four-dimensional hypersurface**!
Needless to say, hard to visualize with eyeballs that perceive only three dimensions!

Functions of Two Variables (Domains)

Definition

The **domain** of $f(x,y)$ is the set of all ordered pairs $(x,y) \in \mathbb{R}^2$ such that $f(x,y)$ is defined:

$$\text{Dom}(f) := \left\{(x,y) \in \mathbb{R}^2 : f(x,y) \text{ is defined}\right\}$$

PROPERTIES OF DOMAINS: Given functions $f(x,y)$ and $g(x,y)$,

- $f(x,y)$ is a **polynomial** in x and $y \implies \text{Dom}(f) = \mathbb{R}^2$
  - Examples of Polynomials: $x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2, x^3y^5, \ldots$
- $\alpha \neq 0 \implies \text{Dom}(\alpha f) = \text{Dom}(f)$
- $\text{Dom}(f+g) = \text{Dom}(f) \cap \text{Dom}(g)$
- $\text{Dom}(f-g) = \text{Dom}(f) \cap \text{Dom}(g)$
- $\text{Dom}(fg) = \text{Dom}(f) \cap \text{Dom}(g)$
- $\text{Dom}(f/g) = [\text{Dom}(f) \cap \text{Dom}(g)] \setminus \{g(x,y) = 0\}$
  [i.e. Find $\text{Dom}(f) \cap \text{Dom}(g)$, then discard points (x,y) where $g(x,y) = 0$]

Level Curves & Contour Plots (Definition)

Definition

Given explicit 3D surface $z = f(x, y)$, the **level curves** are the curves

$$f(x, y) = k, \text{ where } k \in \mathbb{R}$$

projected onto the xy-plane.

Definition

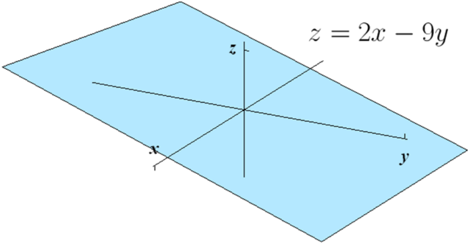
Given implicit 3D surface $F(x, y, z) = 0$, the **level curves** are intersections of the surface with the planes $z = k$ projected onto the xy-plane.

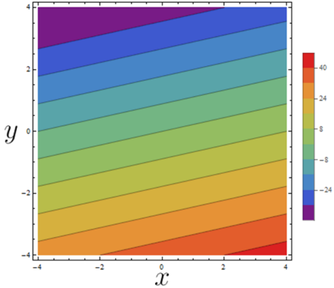
Definition

A **contour plot** is a plot of several level curves on the xy-plane with each region between two level curves shaded certain colors to indicate the z-value at each (x, y)-point.

A conventional color scheme is to let **warm colors** indicate **large values** of z, and **cool colors** indicate **small values** of z.

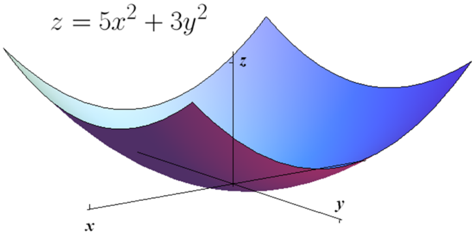
Level Curves in Contour Plots (Bottom Graph)

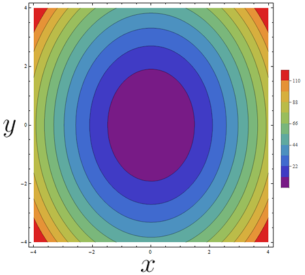




Domain $= \{(x, y) \in \mathbb{R}^2 : 2x - 9y$ is defined$\} = \mathbb{R}^2$

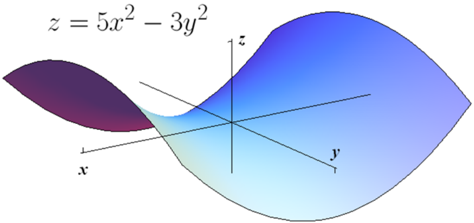
Level Curves in Contour Plots (Bottom Graph)

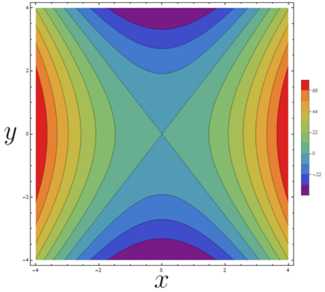




Domain $= \{(x, y) \in \mathbb{R}^2 : 5x^2 + 3y^2$ is defined$\} = \mathbb{R}^2$

Level Curves in Contour Plots (Bottom Graph)

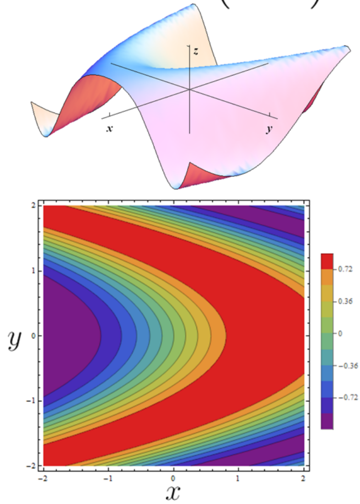




Domain $= \{(x, y) \in \mathbb{R}^2 : 5x^2 - 3y^2$ is defined$\} = \mathbb{R}^2$

Level Curves in Contour Plots (Bottom Graph)

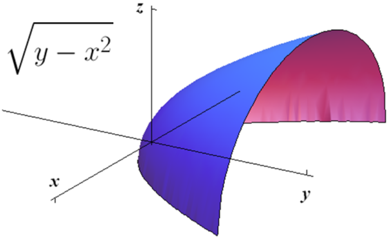
$$f(x,y) = \sin\left(x + y^2\right)$$

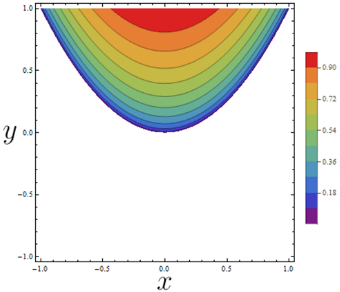


$$\text{Dom}(f) = \left\{(x,y) \in \mathbb{R}^2 : \sin\left(x+y^2\right) \text{ is defined}\right\} = \mathbb{R}^2$$

Level Curves in Contour Plots (Bottom Graph)

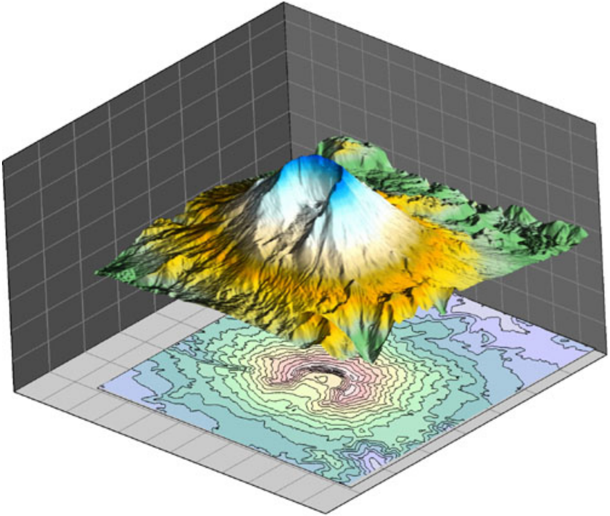
$$f(x,y) = \sqrt{y - x^2}$$



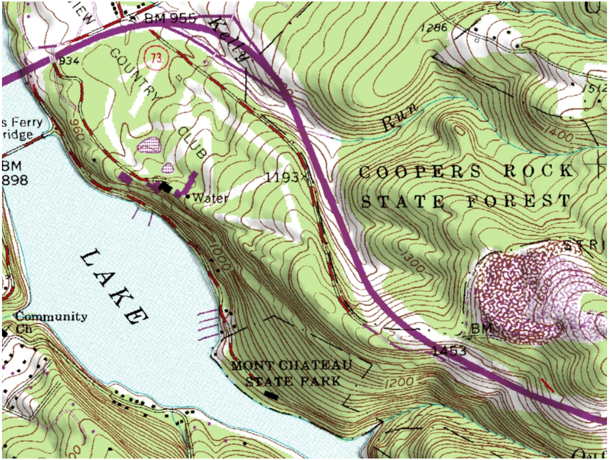


$$\text{Dom}(f) = \left\{(x,y) \in \mathbb{R}^2 : \sqrt{y-x^2} \text{ is defined}\right\} = \left\{(x,y) \in \mathbb{R}^2 : y \geq x^2\right\}$$

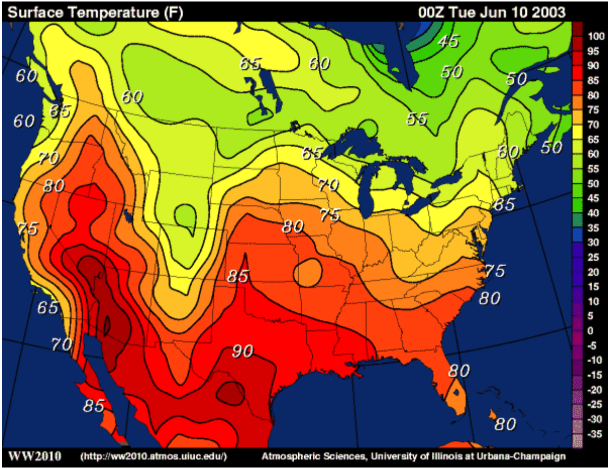
Level Curves (Topographic Maps)



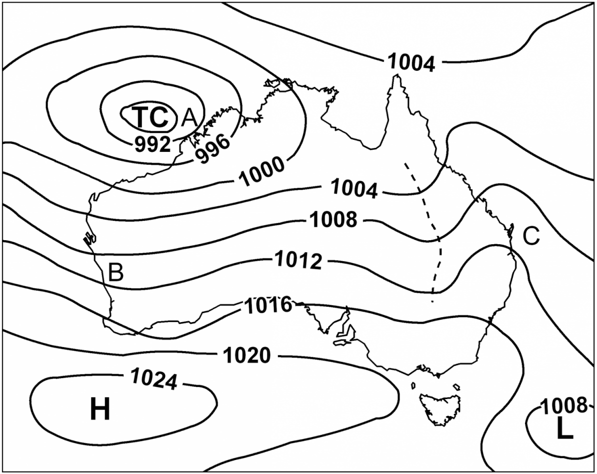
Level Curves (Topographic Maps)



Level Curves (Weather Maps with Isotherms)



Level Curves (Weather Maps with Isobars)



Level Surfaces (Definition)

Definition

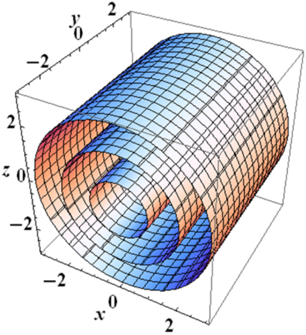
Given explicit 4D hypersurface $w = f(x, y, z)$, the **level surfaces** are the surfaces

$$f(x, y, z) = k, \text{ where } k \in \mathbb{R}$$

projected onto xyz-space.

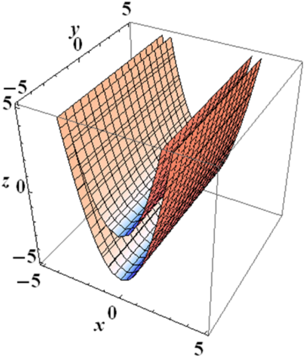
Level Surfaces (Examples)

$$f(x, y, z) = x^2 + z^2$$



Level Surfaces
$x^2 + z^2 = 1$
$x^2 + z^2 = 4$
$x^2 + z^2 = 9$

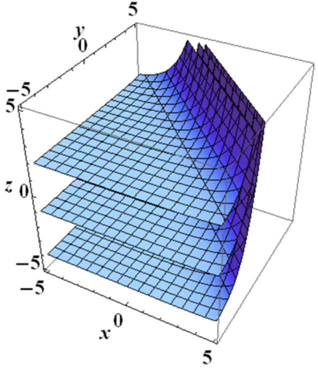
$$f(x, y, z) = x^2 - z$$



Level Surfaces
$x^2 - z = 1$
$x^2 - z = 4$

Level Surfaces (Examples)

$$f(x,y,z) = e^{(x+y-2)} - z$$



Level Surfaces

$$e^{(x+y-2)} - z = -1$$
$$e^{(x+y-2)} - z = 1$$
$$e^{(x+y-2)} - z = 4$$

Fin.