Functions of Several Variables: Limits & Continuity

Calculus III

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TTU

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Limits of 1-Variable Functions (Toolkit)

TASK: Evaluate limit (if it exists): $\lim_{x \to x_0} f(x)$

READ: "The limit as x approaches x_0 of $f(x)$"

- Naïve Substitution (NS)
- Factor Polynomial(s) & cancel like factors
- Rationalize Numerator/Denominator
- Combine Fractions
- Throw a factor "downstairs" e.g. $x^2 \ln x = \dfrac{\ln x}{1/x^2}$
- Change of Variables (CV): Let $u = g(x)$, then limit becomes $\lim_{u \to u_0} h(u)$
- Apply a Trig Identity
- L'Hôpital's Rule (LHOP)
- Find One-Sided Limits first
- Build a Table
- Squeeze Theorem
- δ-ϵ definition of limit – TOO HARD!

Limits of 1-Variable Functions (Infinity)

- Remember, ∞ is **not a real number**, but rather a **symbol** indicating **growth without bound**.
- Similarly, $-\infty$ indicates **decay without bound.**
- However, even though $\pm\infty$ are symbols, they satisfy some arithmetic properties that agree with intuition:
  - $\infty + \infty = \infty \qquad -\infty - \infty = -\infty$
  - $\infty + x = x + \infty = \infty$, where $x \in \mathbb{R}$
  - $-\infty + x = x - \infty = -\infty$, where $x \in \mathbb{R}$
  - $(\infty)(\infty) = \infty, \qquad (-\infty)(-\infty) = \infty$
  - $(-\infty)(\infty) = -\infty, \qquad (\infty)(-\infty) = -\infty$
  - $x > 0 \implies x \cdot \infty = \infty$ and $x \cdot (-\infty) = -\infty$
  - $x < 0 \implies x \cdot \infty = -\infty$ and $x \cdot (-\infty) = \infty$
  - $\infty^n = \infty$, where $n \in \mathbb{N}$
  - $\sqrt[n]{\infty} = \infty$, where $n \in \mathbb{N}$
  - REMARK: $\mathbb{N} := \{1, 2, 3, 4, 5, \cdots\}$ (Set of **Natural Numbers**)

Limits of 1-Variable Functions (Indeterminant Forms)

Recall the **indeterminant forms** from Single Variable Calculus:

$$\frac{0}{0}, \pm\frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$$

Remember that **L'Hôpital's Rule (LHOP)** only works with $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} \overset{NS}{=} \frac{0}{0} \overset{LHOP}{=} \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} \overset{NS}{=} \frac{\infty}{\infty} \overset{LHOP}{=} \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} \overset{NS}{=} -\frac{\infty}{\infty} \overset{LHOP}{=} \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Limits of 2-Variable Functions (Toolkit)

TASK: Evaluate limit (if it exists): $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

READ: "The limit as (x,y) approaches the point (x_0,y_0) of $f(x,y)$"

- Naïve Substitution (NS)
- Factor Polynomial(s) & cancel like factors
- Rationalize Numerator/Denominator
- Combine Fractions
- Throw a factor "downstairs"
- Change of Variables (CV): Let $u = g(x,y)$, then limit becomes $\lim_{u\to u_0} h(u)$
- Convert to Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$
- Apply a Trig Identity – RARELY APPLICABLE!
- L'Hôpital's Rule (LHOP) – USELESS!
- Find One-Sided Limits first – MEANINGLESS!
- Build a Table – TOO TEDIOUS!
- Squeeze Theorem – TOO HARD!
- δ-ϵ definition of limit – TOO HARD!

Limits of 3-Variable Functions (Toolkit)

TASK: Evaluate limit (if it exists): $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$

READ: "The limit as (x,y,z) approaches the point (x_0,y_0,z_0) of $f(x,y,z)$"

- Naïve Substitution (NS)
- Factor Polynomial(s) & cancel like factors
- Rationalize Numerator/Denominator
- Combine Fractions
- Throw a factor "downstairs"
- Change of Variables (CV): Let $u = g(x,y,z)$, then limit becomes $\lim_{u\to u_0} h(u)$
- Convert to Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$ – USELESS!
- Apply a Trig Identity – RARELY APPLICABLE!
- L'Hôpital's Rule (LHOP) – USELESS!
- Find One-Sided Limits first – MEANINGLESS!
- Build a Table – WAY TOO TEDIOUS!
- Squeeze Theorem – TOO HARD!
- δ-ϵ definition of limit – TOO HARD!

Limits of Functions (Naïve Substitution)

WEX 11-2-1: Evaluate $\lim\limits_{t\to4}\sqrt{5+t}$.

$$\lim_{t\to4}\sqrt{5+t}\overset{NS}{=}\sqrt{5+(4)}=\sqrt{9}=\boxed{3}$$

WEX 11-2-2: Evaluate $\lim\limits_{(x,y)\to(2,-5)}\dfrac{1-y}{1+x^2}$.

$$\lim_{(x,y)\to(2,-5)}\frac{1-y}{4+x^2}\overset{NS}{=}\frac{1-(-5)}{4+(2)^2}=\frac{6}{8}=\boxed{\frac{3}{4}}$$

WEX 11-2-3: Evaluate $\lim\limits_{(x,y,z)\to(1,2,3)}\dfrac{\sin(\pi x/2)\cos(\pi y)}{z}$.

$$\lim_{(x,y,z)\to(1,2,3)}\frac{\sin(\pi x/2)\cos(\pi y)}{z}\overset{NS}{=}\frac{\sin(\pi/2)\cos(2\pi)}{3}=\frac{(1)(1)}{3}=\boxed{\frac{1}{3}}$$

WEX 11-2-4: Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{\sin\left(x^2y^3\right)}$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{\sin\left(x^2y^3\right)} \overset{NS}{=} \frac{(0)^2(0)^3}{\sin\left((0)^2(0)^3\right)} = \frac{0}{0} \implies \text{Rewrite/Simplify}$$

CV: Let $u = x^2y^3$. Then, $(x,y) \to (0,0) \implies u \to (0)^2(0)^3 = 0$

$$\implies \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{\sin\left(x^2y^3\right)} \overset{CV}{=} \lim_{u\to 0} \frac{u}{\sin(u)} \overset{NS}{=} \frac{(0)}{\sin(0)} = \frac{0}{0}$$

$$\overset{LHOP}{=} \lim_{u\to 0} \frac{\frac{d}{du}[u]}{\frac{d}{du}[\sin(u)]} = \lim_{u\to 0} \frac{1}{\cos u} \overset{NS}{=} \frac{1}{\cos(0)} = \frac{1}{1} = \boxed{1}$$

Limits of 2-Variable Functions (Polar Coordinates)

<u>WEX 11-2-5:</u> Evaluate $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)\ln\left(x^2+y^2\right)$.

$\overset{NS}{=} \left[(0)^2+(0)^2\right]\ln\left[(0)^2+(0)^2\right] = (0)(-\infty) = 0\cdot\infty \implies$ Rewrite/Simplify

CV: Let $x = r\cos\theta, y = r\sin\theta$. Then, $(x,y)\to(0,0) \implies r\to 0^+$

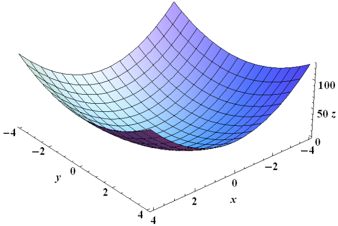
$\overset{CV}{=} \lim\limits_{r\to 0^+} r^2\ln\left(r^2\right) \overset{NS}{=} (0)^2\ln\left[(0)^2\right] = (0)(-\infty) = 0\cdot\infty \implies$ Rewrite/Simplify

Throw factor downstairs: $\lim\limits_{r\to 0^+} r^2\ln\left(r^2\right) = \lim\limits_{r\to 0^+} \dfrac{\ln\left(r^2\right)}{1/r^2} \overset{NS}{=} \dfrac{\ln\left[(0)^2\right]}{1/(0)^2} = \dfrac{-\infty}{\infty}$

$$\overset{LHOP}{=} \lim_{r\to 0^+} \frac{\frac{d}{dr}\left[\ln\left(r^2\right)\right]}{\frac{d}{dr}\left[1/r^2\right]} = \lim_{r\to 0^+} \frac{\frac{2}{r}}{-\frac{2}{r^3}} = \lim_{r\to 0^+}\left(-r^2\right) \overset{NS}{=} -(0)^2 = \boxed{0}$$

Limits of 2-Variable Functions (Existence)

Consider the limit $\lim\limits_{(x,y)\to(0,0)} \left(5x^2 + 3y^2\right)$



Then $\lim\limits_{(x,y)\to(0,0)} \left(5x^2 + 3y^2\right) \overset{NS}{=} 5(0)^2 + 3(0)^2 = \boxed{0}$

Therefore, the limit **exists**, meaning no matter what path (curve) is chosen to approach $(0,0)$, the limit value (z-coordinate) **always** approaches 0.

But, if the function is complicated enough where the usual techniques don't work, one has to prove existence of the limit using the **Squeeze Theorem** and/or δ-ϵ **definiton of a limit**, both of which are very, very hard to use! (Such a scenario is covered in **Advanced Calculus**.)

Limits of 2-Variable Functions (Existence)

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Even if the limit values agree when approached from hundreds (or even billions) of different paths, it does not necessary mean that the limit exists!

All it takes is for the limit values approached from **two different paths** to **not agree** to show that a limit **does not exist**.

Limits of 2-Variable Functions (Non-Existence)

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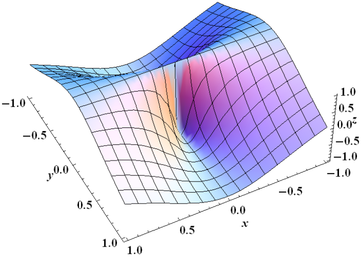
WEX 11-2-6: Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.



Move along y-axis $(x = 0)$: $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} \frac{(0)^2 - y^2}{(0)^2 + y^2} = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

Approach along x-axis $(y = 0)$: $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - (0)^2}{x^2 + (0)^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$

The limit values do not agree using two different paths to $(0, 0)$.
Therefore, the limit **D**oes **N**ot **E**xist (**DNE**).

Continuity of Functions of Two Var's (Definition)

Definition

$f(x,y)$ is **continuous at point** (x_0,y_0) if all three conditions hold:

$$f(x_0,y_0) \text{ exists}$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ exists}$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Definition

$f(x,y)$ is **continuous on set** $S \subseteq \mathbb{R}^2 \iff f$ is continuous at each point in S.

NOTATION: $f \in C(S)$

- **Polynomials** in two variables are **continuous everywhere** $(\text{on } \mathbb{R}^2)$.
  - Examples of Polynomials: $x, y, x^2y, xy, x^2 - xy + y^2, \ldots$

Continuity of Functions of Three Var's (Definition)

Definition

$f(x,y,z)$ is **continuous at point** (x_0,y_0,z_0) if all three conditions hold:

$$f(x_0,y_0,z_0) \text{ exists}$$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) \text{ exists}$$

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$$

Definition

$f(x,y,z)$ is **continuous on set** $S \subseteq \mathbb{R}^3 \iff f$ is continuous at each point in S.

NOTATION: $f \in C(S)$

- **Polynomials** in three variables are **continuous everywhere** $(\text{on } \mathbb{R}^3)$.
  - Examples of Polynomials: $x, y, x^2y, xyz, y^2z^4, x^3 + xyz - xy - yz + y^3, \ldots$

Fin.