Functions of Several Variables: Partial Derivatives

Calculus III

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PART I:

PARTIAL DERIVATIVES

1^{st}-Order Partial Derivatives of $f(x,y)$

Definition

Given a function of two variables $f(x,y)$:

(**Partial Derivative** of f w.r.t. x) $$\frac{\partial f}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

(**Partial Derivative** of f w.r.t. y) $$\frac{\partial f}{\partial y} := \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

w.r.t $\equiv$ "with respect to"

NOTATION:

"partial f partial x"	$\frac{\partial f}{\partial x}$	$\frac{\partial}{\partial x}\left[f(x,y)\right]$	f_x
"partial f partial y"	$\frac{\partial f}{\partial y}$	$\frac{\partial}{\partial y}\left[f(x,y)\right]$	f_y

WEX 11-3-1: Let $f(x,y) = xy$. Compute $\dfrac{\partial f}{\partial x}$ using the definition.

$$\frac{\partial f}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)y - xy}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{xy + y\Delta x - xy}{\Delta x} = \lim_{\Delta x \to 0} \frac{y\Delta x}{\Delta x} = \lim_{\Delta x \to 0} y = \boxed{y}$$

WEX 11-3-2: Let $f(x,y) = xy$. Compute $\dfrac{\partial f}{\partial y}$ using the definition.

$$\frac{\partial f}{\partial y} := \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x(y+\Delta y) - xy}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{xy + x\Delta y - xy}{\Delta y} = \lim_{\Delta y \to 0} \frac{x\Delta y}{\Delta y} = \lim_{\Delta y \to 0} x = \boxed{x}$$

1^{st}-Order Partial Derivatives of Multivariable Functions

Just as in Calculus I, using the definition of a partial derivative is, in general, tedious at best and untenable at worst!

Fortunately, there's an easier procedure:

> Use the ordinary derivative rules from Calculus I,
>
> but **treat the other independent variable(s) as <u>constants</u>**.

Review of Ordinary Derivative Rules from Calculus I

DERIVATIVE RULE	FORMULA	REMARKS
Constant Rule	$\frac{d}{dx}\left[k\right] = 0$	$k \in \mathbb{R}$
Power Rule	$\frac{d}{dx}\left[x^k\right] = kx^{k-1}$	$k \in \mathbb{R}$
Constant Multiple Rule	$\frac{d}{dx}\left[kf(x)\right] = k\frac{df}{dx}$	$k \in \mathbb{R}$
Sum/Difference Rule	$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{df}{dx} \pm \frac{dg}{dx}$	
Product Rule	$\frac{d}{dx}\left[f(x)g(x)\right] = g(x)\frac{df}{dx} + f(x)\frac{dg}{dx}$	
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{\left[g(x)\right]^2}$	$g(x) \neq 0$
Chain Rule (usual form)	$\frac{d}{dx}\left[f[g(x)]\right] = f'[g(x)]g'(x)$	$f \circ g \equiv f[g(x)]$

QUOTIENT RULE: ”Lo D-Hi Minus Hi D-Lo All Over Lo-Squared”

Review of Ordinary Derivative Rules from Calculus I

- $\frac{d}{dx}[\sin x] = \cos x$ $\qquad \frac{d}{dx}[\cos x] = -\sin x$
- $\frac{d}{dx}[\tan x] = \sec^2 x$ $\qquad \frac{d}{dx}[\cot x] = -\csc^2 x$
- $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$
- $\frac{d}{dx}[e^x] = e^x$ $\qquad \frac{d}{dx}[\ln x] = \frac{1}{x}$
- $\frac{d}{dx}[a^x] = (\ln a)a^x$ $\quad (a > 0)$ $\qquad \frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)} \cdot \frac{1}{x}$ $\quad (a > 0 \text{ and } a \neq 1)$
- $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}[\text{arccot}\, x] = -\frac{1}{1+x^2}$
- $\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{|x|\sqrt{x^2-1}}$ $\qquad \frac{d}{dx}[\text{arccsc}\, x] = -\frac{1}{|x|\sqrt{x^2-1}}$

1^{st}-Order Partial Derivatives of $f(x, y)$

WEX 11-3-3: Let $f(x, y) = xy$. Compute $\frac{\partial f}{\partial x}$.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\Big[xy\Big] && \Big(\text{Going forward, treat } y \text{ as a constant}\Big) \\ &= y\frac{\partial}{\partial x}\Big[x\Big] && \Big(\text{Constant Multiple Rule}\Big) \\ &= y(1) && \Big(\text{Power Rule}\Big) \\ &= \boxed{y} \end{aligned}$$

WEX 11-3-4: Let $f(x, y) = xy$. Compute $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\Big[xy\Big] = \Big(\text{Now treat } x \text{ as a constant}\Big) = x\frac{\partial}{\partial y}\Big[y\Big] = x(1) = \boxed{x}$$

1^{st}-Order Partials of $f(x, y)$ (Interpretation)

Recall the interpretation of the 1^{st}-order **ordinary derivative** of $f(x)$:

$\frac{df}{dx}$ measures the (instantaneous) rate of change of f as x changes.

Now, here's the interpretation of the 1^{st}-Order Partials of $f(x, y)$:

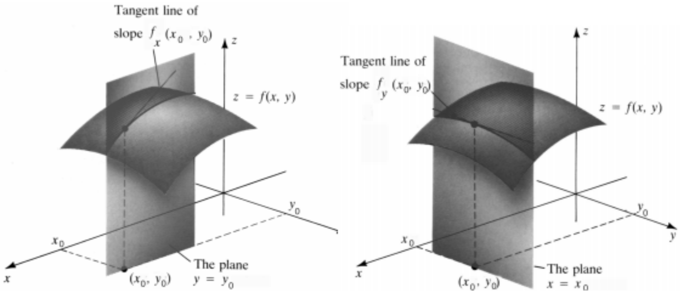
$\frac{\partial f}{\partial x}$ measures the rate of change of f as x changes, holding y constant.
$\frac{\partial f}{\partial y}$ measures the rate of change of f as y changes, holding x constant.

WEX 11-3-5: In physics, **Ohm's Law** is $E = IR$, where $E \equiv$ Voltage, $I \equiv$ Current, $R \equiv$ Resistance.

$\frac{\partial E}{\partial I}$ measures the rate of change of Voltage as Current changes, holding Resistance constant.

$\frac{\partial E}{\partial R}$ measures the rate of change of Voltage as Resistance changes, holding Current constant.

1^{st}-Order Partials of $f(x,y)$ (Geometric Interpretation)



Geometrically:

- $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ is the **slope** of the **tangent line** to the **intersection** of surface f with **plane** $y = y_0$.
- $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ is the **slope** of the **tangent line** to the **intersection** of surface f with **plane** $x = x_0$.

1^{st}-Order Partial Derivatives of $f(x,y,z)$

Definition

Given a function of three variables $f(x,y,z)$:

(**Partial Derivative** of f w.r.t. x) $$\frac{\partial f}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

(**Partial Derivative** of f w.r.t. y) $$\frac{\partial f}{\partial y} := \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$$

(**Partial Derivative** of f w.r.t. z) $$\frac{\partial f}{\partial z} := \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z}$$

NOTATION:

"partial f partial x"	$\frac{\partial f}{\partial x}$	$\frac{\partial}{\partial x}\left[f(x,y,z)\right]$	f_x
"partial f partial y"	$\frac{\partial f}{\partial y}$	$\frac{\partial}{\partial y}\left[f(x,y,z)\right]$	f_y
"partial f partial z"	$\frac{\partial f}{\partial z}$	$\frac{\partial}{\partial z}\left[f(x,y,z)\right]$	f_z

1^{st}-Order Partial Derivatives of $f(x, y, z)$

WEX 11-3-6: Let $f(x, y, z) = xyz$. Compute $\dfrac{\partial f}{\partial x}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xyz\right] = \Big(\text{Now treat } y \text{ \& } z \text{ as constants}\Big) = yz\frac{\partial}{\partial x}\left[x\right] = yz(1) = \boxed{yz}$$

WEX 11-3-7: Let $f(x, y, z) = xyz$. Compute $\dfrac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xyz\right] = \Big(\text{Now treat } x \text{ \& } z \text{ as constants}\Big) = xz\frac{\partial}{\partial y}\left[y\right] = xz(1) = \boxed{xz}$$

WEX 11-3-8: Let $f(x, y, z) = xyz$. Compute $\dfrac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[xyz\right] = \Big(\text{Now treat } x \text{ \& } y \text{ as constants}\Big) = xy\frac{\partial}{\partial z}\left[z\right] = xy(1) = \boxed{xy}$$

1^{st}-Order Partials of $f(x,y,z)$ (Interpretation)

Recall the interpretation of the 1^{st}-order **ordinary derivative** of $f(x)$:

$\frac{df}{dx}$ measures the (instantaneous) rate of change of f as x changes.

Now, here's the interpretation of the 1^{st}-Order Partials of $f(x,y,z)$:

$\frac{\partial f}{\partial x}$ measures the rate of change of f as x changes, holding y & z constant.
$\frac{\partial f}{\partial y}$ measures the rate of change of f as y changes, holding x & z constant.
$\frac{\partial f}{\partial z}$ measures the rate of change of f as z changes, holding x & y constant.

2^{nd}-Order Partial Derivatives of $f(x, y)$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &:= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] &= (f_x)_x &= f_{xx} \\
\frac{\partial^2 f}{\partial y \partial x} &:= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] &= (f_x)_y &= f_{xy} \\
\frac{\partial^2 f}{\partial x \partial y} &:= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] &= (f_y)_x &= f_{yx} \\
\frac{\partial^2 f}{\partial y^2} &:= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] &= (f_y)_y &= f_{yy}
\end{aligned}
$$

REMARK: $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are often called the **mixed** 2^{nd} **partials** of f.

2^{nd}-Order Partial Derivatives of $f(x, y, z)$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &:= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] &= (f_x)_x &= f_{xx} \\
\frac{\partial^2 f}{\partial y^2} &:= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] &= (f_y)_y &= f_{yy} \\
\frac{\partial^2 f}{\partial z^2} &:= \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z}\right] &= (f_z)_z &= f_{zz} \\
\frac{\partial^2 f}{\partial y \partial x} &:= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] &= (f_x)_y &= f_{xy} \\
\frac{\partial^2 f}{\partial x \partial y} &:= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] &= (f_y)_x &= f_{yx} \\
\frac{\partial^2 f}{\partial z \partial x} &:= \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial x}\right] &= (f_x)_z &= f_{xz} \\
\frac{\partial^2 f}{\partial x \partial z} &:= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial z}\right] &= (f_z)_x &= f_{zx} \\
\frac{\partial^2 f}{\partial z \partial y} &:= \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial y}\right] &= (f_y)_z &= f_{yz} \\
\frac{\partial^2 f}{\partial y \partial z} &:= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right] &= (f_z)_y &= f_{zy}
\end{aligned}
$$

3^{rd}-Order Partial Derivatives of $f(x, y)$

$$\frac{\partial^3 f}{\partial x^3} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xxx}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xxy}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial x} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xyx}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xyy}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x \partial y}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]\right] = f_{yxx}$$

$$\frac{\partial^3 f}{\partial y \partial x \partial y} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial x \partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]\right] = f_{yxy}$$

$$\frac{\partial^3 f}{\partial x \partial y^2} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\right] = f_{yyx}$$

$$\frac{\partial^3 f}{\partial y^3} := \frac{\partial}{\partial y}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\right] = f_{yyy}$$

3^{rd}-Order Partial Derivatives of $f(x, y, z)$

There are 27 3^{rd}-order partials – too many to list! Here are seven of them:

$$\frac{\partial^3 f}{\partial x^3} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xxx}$$

$$\frac{\partial^3 f}{\partial z^3} := \frac{\partial}{\partial z}\left[\frac{\partial^2 f}{\partial z^2}\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z}\right]\right] = f_{zzz}$$

$$\frac{\partial^3 f}{\partial z \partial y^2} := \frac{\partial}{\partial z}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\right] = f_{yyz}$$

$$\frac{\partial^3 f}{\partial x^2 \partial z} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial x \partial z}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial z}\right]\right] = f_{zxx}$$

$$\frac{\partial^3 f}{\partial x \partial z \partial x} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial z \partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xzx}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} := \frac{\partial}{\partial x}\left[\frac{\partial^2 f}{\partial y \partial z}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial z}\right]\right] = f_{zyx}$$

$$\frac{\partial^3 f}{\partial z \partial y \partial x} := \frac{\partial}{\partial z}\left[\frac{\partial^2 f}{\partial y \partial x}\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]\right] = f_{xyz}$$

PART II:

TOTAL DIFFERENTIALS & ERROR ESTIMATION

Total Differentials

Definition

Given $f(x,y)$. Then, **Total differential** $df := \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

Given $f(x,y,z)$. Then, **Total differential** $df := \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy + \dfrac{\partial f}{\partial z}dz$

WEX 11-3-9: Find the total differential of $f(x,y) = \sin(xy)$.

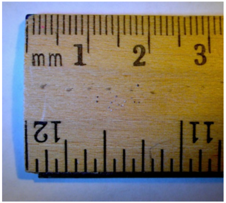
First, find the 1^{st}-order partials of f:

$$f_x = \frac{\partial}{\partial x}\Big[\sin(xy)\Big] = \Big(\text{Treat } y \text{ as a constant}\Big) = \cos(xy)\frac{\partial}{\partial x}\Big[xy\Big] = y\cos(xy)$$

$$f_y = \frac{\partial}{\partial y}\Big[\sin(xy)\Big] = \Big(\text{Treat } x \text{ as a constant}\Big) = \cos(xy)\frac{\partial}{\partial y}\Big[xy\Big] = x\cos(xy)$$

$$\implies df = f_x\,dx + f_y\,dy = \boxed{y\cos(xy)\,dx + x\cos(xy)\,dy}$$

Measurements are Never 100% Accurate



If there's error in measuring x and y, how can one **estimate** the error in $f(x,y)$?
If there's error in measuring x,y,z, how can one **estimate** the error in $g(x,y,z)$?

Linear Error Approximation

Definition

Given $f(x, y)$ and "small" errors $\Delta x, \Delta y$ in x, y.

Then, **Linear error** $\Delta f \approx \dfrac{\partial f}{\partial x}\Delta x + \dfrac{\partial f}{\partial y}\Delta y$

Given $f(x, y, z)$ and "small" errors $\Delta x, \Delta y, \Delta z$ in x, y, z.

Then, **Linear error** $\Delta f \approx \dfrac{\partial f}{\partial x}\Delta x + \dfrac{\partial f}{\partial y}\Delta y + \dfrac{\partial f}{\partial z}\Delta z$

REMARK: The error is "linear" in the sense that the expression contains no powers of Δx or Δy.

Linear Error Approximation

WEX 11-3-10: Given $f(x,y) = e^{xy}$ and "small" errors $\Delta x, \Delta y$ in x, y, find the linear error Δf.

First, find the 1^{st}-order partials of f:

$$f_x = \frac{\partial}{\partial x}\left[e^{xy}\right] = \left(\text{Treat } y \text{ as a constant}\right) = e^{xy}\frac{\partial}{\partial x}\left[xy\right] = ye^{xy}$$

$$f_y = \frac{\partial}{\partial y}\left[e^{xy}\right] = \left(\text{Treat } x \text{ as a constant}\right) = e^{xy}\frac{\partial}{\partial y}\left[xy\right] = xe^{xy}$$

Next, find the total differential df:

$$df = f_x\,dx + f_y\,dy = ye^{xy}\,dx + xe^{xy}\,dy$$

Finally, find the linear error Δf:

$$\boxed{\Delta f \approx ye^{xy}\Delta x + xe^{xy}\Delta y}$$

Notation for Continuous Derivatives (Calculus I)

Recall the notation for a **continuous function** f on a set S: $\quad f \in C(S)$

Definition

Given $f(x)$ and set $S \subseteq \mathbb{R}$. Then:

$$f \in C^1(S) \iff f, f' \in C(S)$$

$$f \in C^2(S) \iff f, f', f'' \in C(S)$$

Notation for Continuous Partial Derivatives

Recall the notation for a **continuous function** f on a set S: $\quad f \in C(S)$

Definition

Given $f(x,y)$ and set $S \subseteq \mathbb{R}^2$. Then:

$$f \in C^{(1,1)}(S) \iff f, f_x, f_y \in C(S)$$

$$f \in C^{(2,2)}(S) \iff f, f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx} \in C(S)$$

Definition

Given $f(x,y,z)$ and set $S \subseteq \mathbb{R}^3$. Then:

$$f \in C^{(1,1,1)}(S) \iff f, f_x, f_y, f_z \in C(S)$$

$$f \in C^{(2,2,2)}(S) \iff f, f_x, f_y, f_z, f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yx}, f_{xz}, f_{zx}, f_{yz}, f_{zy} \in C(S)$$

Differentiability

Theorem

*(**Sufficient Condition** for Differentiability)*

Given $f(x,y)$, *then* f *is **differentiable** on set* $S \subseteq \mathbb{R}^2$ *if* $f \in C^{(1,1)}(S)$

Given $f(x,y,z)$, *then* f *is **differentiable** on set* $S \subseteq \mathbb{R}^3$ *if* $f \in C^{(1,1,1)}(S)$

REMARK: To learn the **necessary condition(s)** for differentiability of $f(x,y)$ or $f(x,y,z)$, take **Advanced Calculus**.

Fin.