Functions of Several Variables: Chain Rules

Calculus III

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TTU

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PART I:
MULTIVARIABLE CHAIN RULES

1-1 Chain Rule (from Calculus I)

Let $y = f(x) \in C^1$ where $x = g(t) \in C^1$.

Determine the chain rule formula for $\frac{dy}{dt}$.

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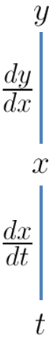
First, sketch the **dependency tree** for y:



1-1 Chain Rule (from Calculus I)

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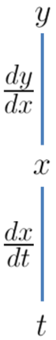
Determine the chain rule formula for $\dfrac{dy}{dt}$.

"Slide down" the tree:



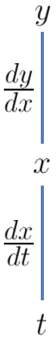
Then, $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

1-1 Chain Rule (from Calculus I)

Proposition

Let $y = f(x) \in C^1$ *where* $x = g(t) \in C^1$. *Then:*

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$



"1-1" means 1 **intermediate variable** (x) and 1 **independent variable** (t).

1-2 Chain Rule

Let $z = f(x) \in C^1$ where $x = g(s,t) \in C^{(1,1)}$.

Determine the chain rule formulas for $\dfrac{\partial z}{\partial s}$ & $\dfrac{\partial z}{\partial t}$.

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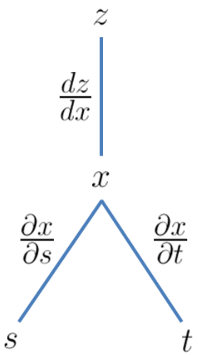

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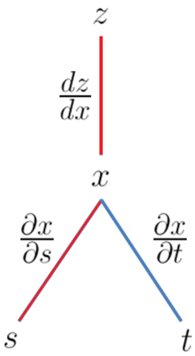
Let $z = f(x) \in C^1$ where $x = g(s,t) \in C^{(1,1)}$.

Determine the chain rule formulas for $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

"Slide down" the branch of the tree that has s as its bottom node (in red):



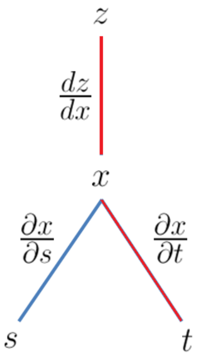
Then, $\frac{\partial z}{\partial s} = \frac{dz}{dx}\frac{\partial x}{\partial s}$

1-2 Chain Rule

Let $z = f(x) \in C^1$ where $x = g(s,t) \in C^{(1,1)}$.

Determine the chain rule formulas for $\dfrac{\partial z}{\partial s}$ & $\dfrac{\partial z}{\partial t}$.

"Slide down" the branch of the tree that has t as its bottom node (in red):



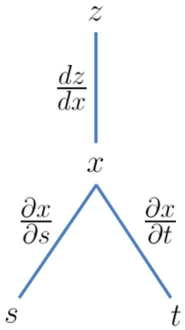
Then, $\dfrac{\partial z}{\partial t} = \dfrac{dz}{dx}\dfrac{\partial x}{\partial t}$

1-2 Chain Rule

Proposition

Let $z = f(x) \in C^1$ *where* $x = g(s,t) \in C^{(1,1)}$*.* *Then:*

$$\frac{\partial z}{\partial s} = \frac{dz}{dx}\frac{\partial x}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{dz}{dx}\frac{\partial x}{\partial t}$$



"1-2" means 1 **intermediate variable** (x) and 2 **independent var's** (s,t).

2-1 Chain Rule

Let $z = f(x,y) \in C^{(1,1)}$ where $x = g(t) \in C^1$ and $y = h(t) \in C^1$.

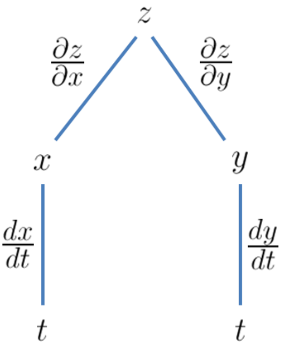
Determine the chain rule formula for $\dfrac{dz}{dt}$.

2-1 Chain Rule

Let $z = f(x, y) \in C^{(1,1)}$ where $x = g(t) \in C^1$ and $y = h(t) \in C^1$.

Determine the chain rule formula for $\dfrac{dz}{dt}$.

First, sketch the **dependency tree** for z:



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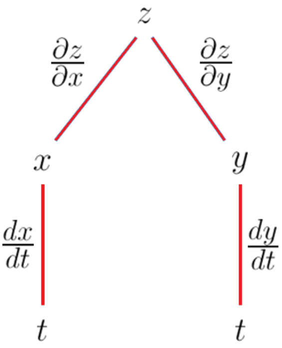
Let $z = f(x,y) \in C^{(1,1)}$ where $x = g(t) \in C^1$ and $y = h(t) \in C^1$.

Determine the chain rule formula for $\frac{dz}{dt}$.

"Slide down" each branch of the tree that has t as its bottom node (in red).



Then, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

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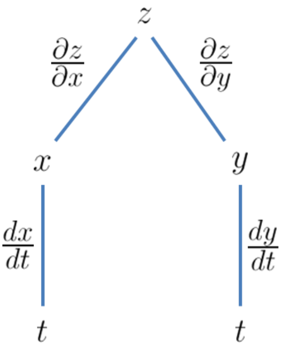

"2-1" means 2 **intermediate var's** (x,y) and 1 **independent variable** (t).

2-2 Chain Rule

Let $z = f(x, y) \in C^{(1,1)}$ where $x = g(s, t) \in C^{(1,1)}$ and $y = h(s, t) \in C^{(1,1)}$.

Determine the chain rule formula for $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

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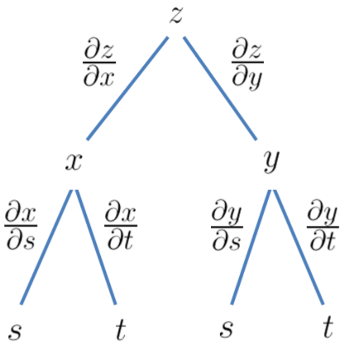

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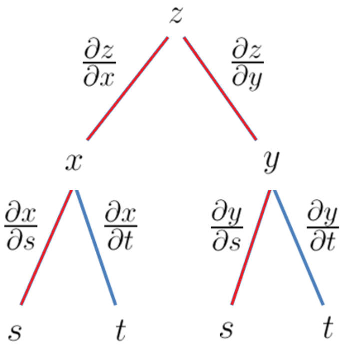
Determine the chain rule formula for $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

"Slide down" each branch of the tree that has s as its bottom node (in red).



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

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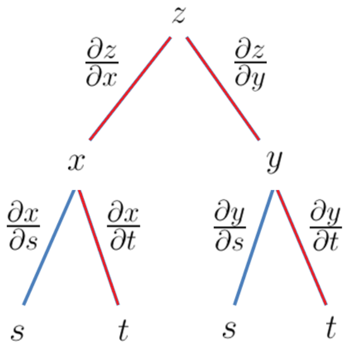
Let $z = f(x, y) \in C^{(1,1)}$ where $x = g(s, t) \in C^{(1,1)}$ and $y = h(s, t) \in C^{(1,1)}$.

Determine the chain rule formula for $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

"Slide down" each branch of the tree that has t as its bottom node (in red).



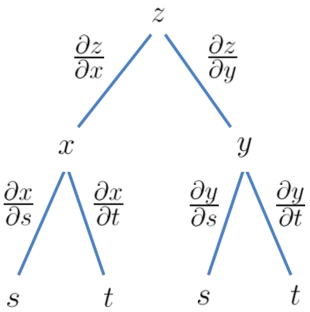
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

2-2 Chain Rule

Proposition

Let $z = f(x,y) \in C^{(1,1)}$ *where* $x = g(s,t) \in C^{(1,1)}$ *and* $y = h(s,t) \in C^{(1,1)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



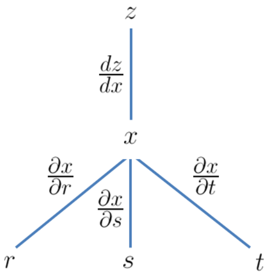
"2-2" means 2 **intermediate var's** (x,y) and 2 **independent var's** (s,t).

1-3 Chain Rule

Proposition

Let $z = f(x) \in C^1$ *where* $x = g(r,s,t) \in C^{(1,1,1)}$. *Then:*

$$\frac{\partial z}{\partial r} = \frac{dz}{dx}\frac{\partial x}{\partial r} \qquad \frac{\partial z}{\partial s} = \frac{dz}{dx}\frac{\partial x}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{dz}{dx}\frac{\partial x}{\partial t}$$



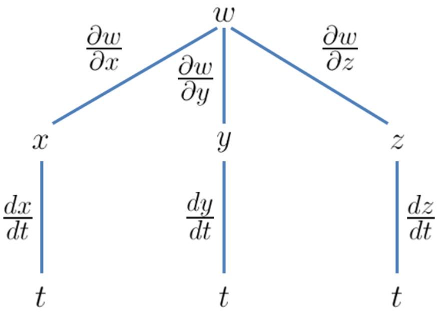
"1-3" means 1 **intermediate variable** (x) and 3 **independent var's** (r,s,t).

3-1 Chain Rule

Proposition

Let $w = f(x,y,z) \in C^{(1,1,1)}$ *s.t.* $x = g(t) \in C^1$, $y = h(t) \in C^1$, $z = p(t) \in C^1$. *Then:*

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$



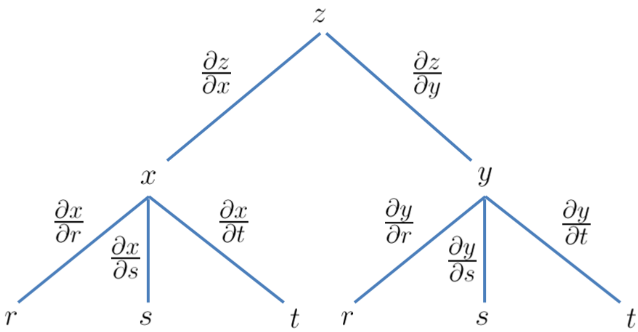
"3-1" means 3 **intermediate var's** (x,y,z) and 1 **independent variable** (t).

2-3 Chain Rule

Proposition

Let $z = f(x, y) \in C^{(1,1)}$ *s.t.* $x = g(r, s, t) \in C^{(1,1,1)}$, $y = h(r, s, t) \in C^{(1,1,1)}$. *Then:*

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



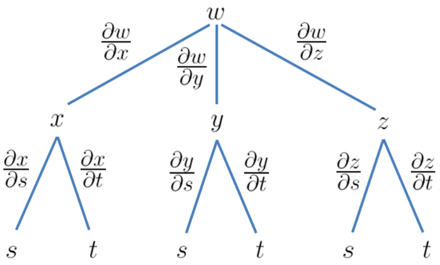
"2-3" means 2 **intermediate var's** (x, y) and 3 **independent var's** (r, s, t).

3-2 Chain Rule

Proposition

Let $w = f(x, y, z) \in C^{(1,1,1)}$ *s.t.* $x = g(s, t)$, $y = h(s, t)$, $z = p(s, t)$. *Then:*

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$



"3-2" means 3 **intermediate var's** (x, y, z) and 2 **independent var's** (s, t).

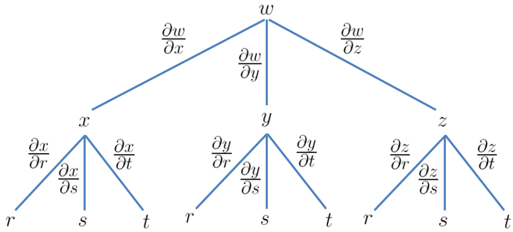
3-3 Chain Rule

Proposition

Let $w = f(x,y,z) \in C^{(1,1,1)}$ *s.t.* $x = g(r,s,t)$, $y = h(r,s,t)$, $z = p(r,s,t)$. *Then:*

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$



"3-3" means 3 **intermediate var's** (x,y,z) and 3 **independent var's** (r,s,t).

PART II:

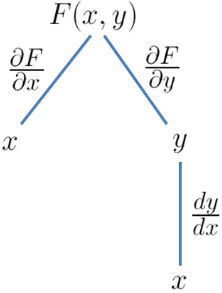
IMPLICIT DIFFERENTIATION

Implicit Differentiation (2-Variable Function)

Proposition

Let $F(x,y) = 0$ s.t. $F \in C^{(1,1)}$ and y is implicitly a function of x. Then:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}, \text{ provided } F_y \neq 0$$



$$
\begin{aligned}
& dF/dx = 0 \\
\implies & \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \\
\implies & \frac{\partial F}{\partial y}\frac{dy}{dx} = -\frac{\partial F}{\partial x} \\
\implies & \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}
\end{aligned}
$$

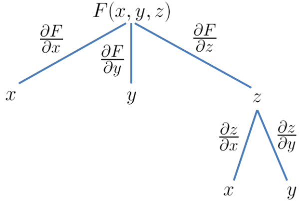
Implicit Differentiation (3-Variable Function)

Proposition

Let $F(x,y,z) = 0$ *s.t.* $F \in C^{(1,1,1)}$ *and* z *is implicitly a function of* (x,y). *Then:*

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \text{ provided } F_z \neq 0$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \text{ provided } F_z \neq 0$$



$$\partial F / \partial x = 0$$

$$\implies \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\implies \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}$$

$$\implies \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$$

$\Big($Similarly for $\partial z / \partial y\Big)$

Related Rates....Revisited

WEX 11-5-1:

A rectangle is changing in such a way that its length ℓ is decreasing at a rate of 4 mm/min and its width w is increasing at a rate of 10 mm/min.
At what rates are its area & perimeter changing when the length is 5 mm and the width is 12 mm?

1^{st}, realize that the **independent variable** is **time** (t).

2^{nd}, recall the formulas for area & perimeter: $A = \ell w \qquad P = 2\ell + 2w$.

3^{rd}, use "2-1" Chain Rule: $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial \ell}\dfrac{d\ell}{dt} + \dfrac{\partial A}{\partial w}\dfrac{dw}{dt} \qquad \dfrac{dP}{dt} = \dfrac{\partial P}{\partial \ell}\dfrac{d\ell}{dt} + \dfrac{\partial P}{\partial w}\dfrac{dw}{dt}$

4^{th}, extract info: $\ell = 5$, $w = 12$, $\dfrac{d\ell}{dt} = -4$, $\dfrac{dw}{dt} = 10$.

5^{th}, compute the partials: $\dfrac{\partial A}{\partial \ell} = w = 12, \quad \dfrac{\partial A}{\partial w} = \ell = 5 \qquad \dfrac{\partial P}{\partial \ell} = 2, \quad \dfrac{\partial P}{\partial w} = 2$

$$\frac{dA}{dt} = \frac{\partial A}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt} = \Big(12\Big)\Big(-4\Big) + \Big(5\Big)\Big(10\Big) = \boxed{2 \text{ mm}^2/\text{min}}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial P}{\partial w}\frac{dw}{dt} = \Big(2\Big)\Big(-4\Big) + \Big(2\Big)\Big(10\Big) = \boxed{12 \text{ mm/min}}$$

Fin

Fin.