Functions of Several Variables: Gradients

Calculus III

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PART I:

GRADIENTS & DIRECTIONAL DERIVATIVES

Function of Several Variables (Gradient)

Definition

Let $f(x,y) \in C^{(1,1)}$. Then the **gradient** of f is a vector in $\mathbb{R}^2$ given by:

$$\text{grad}\, f \equiv \nabla f(x,y) \equiv \nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = f_x \widehat{\mathbf{i}} + f_y \widehat{\mathbf{j}}$$

Definition

Let $f(x,y,z) \in C^{(1,1,1)}$. Then the **gradient** of f is a vector in $\mathbb{R}^3$ given by:

$$\text{grad}\, f \equiv \nabla f(x,y,z) \equiv \nabla f := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = f_x \widehat{\mathbf{i}} + f_y \widehat{\mathbf{j}} + f_z \widehat{\mathbf{k}}$$

READ: "grad f" or "del f"

REMARK: The gradient encapsulates all the 1^{st}-order partials into a vector.

Gradient (Computation)

WEX 11-6-1: Given $f(x,y) = xy$, compute ∇f and $\nabla f(4,-7)$.

First, find all 1^{st}-order partials: $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\Big[xy\Big] = y, \quad \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\Big[xy\Big] = x$

$$\implies \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \boxed{\langle y, x\rangle} \implies \nabla f(4,-7) = \boxed{\langle -7, 4\rangle}$$

WEX 11-6-2: Let $g(x,y,z) = xy + yz + xz$, find ∇g and $\nabla g(1,2,-3)$.

First, find all 1^{st}-order partials: $g_x = y + z, \quad g_y = x + z, \quad g_z = y + x$

$$\implies \nabla g = \langle g_x, g_y, g_z\rangle = \boxed{\langle y+z, x+z, y+x\rangle}$$

$$\implies \nabla g(1,2,-3) = \langle (2)+(-3), (1)+(-3), (2)+(1)\rangle = \boxed{\langle -1, -2, 3\rangle}$$

Gradient (Properties)

Let $f, g \in C^{(1,1)}$ or $f, g \in C^{(1,1,1)}$. Then:

RULE	FORM	REMARK(S)
Constant Rule	$\nabla k = \vec{\mathbf{0}}$	$k \in \mathbb{R}$
Constant Multiple Rule	$\nabla(kf) = k\nabla f$	$k \in \mathbb{R}$
Sum/Diff Rule	$\nabla(f \pm g) = \nabla f \pm \nabla g$	
Product Rule	$\nabla(fg) = f\nabla g + g\nabla f$	
Quotient Rule	$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$	$g \neq 0$
Power Rule	$\nabla\left(f^n\right) = nf^{n-1}\nabla f$	

QUOTIENT RULE: ”Lo Grad-Hi Minus Hi Grad-Lo All Over Lo Squared”

Gradient (Proof of Sum Rule)

PROOF OF SUM RULE:

Let $f, g \in C^{(1,1)}$. Then:

$$\begin{aligned}
\nabla(f+g) &= \left\langle \frac{\partial}{\partial x}\left[f+g\right], \frac{\partial}{\partial y}\left[f+g\right] \right\rangle \\
&= \langle f_x + g_x, f_y + g_y \rangle \\
&= \langle f_x, f_y \rangle + \langle g_x, g_y \rangle \\
&= \nabla f + \nabla g
\end{aligned}$$

Let $f, g \in C^{(1,1,1)}$. Then:

$$\begin{aligned}
\nabla(f+g) &= \left\langle \frac{\partial}{\partial x}\left[f+g\right], \frac{\partial}{\partial y}\left[f+g\right], \frac{\partial}{\partial z}\left[f+g\right] \right\rangle \\
&= \langle f_x + g_x, f_y + g_y, f_z + g_z \rangle \\
&= \langle f_x, f_y, f_z \rangle + \langle g_x, g_y, g_z \rangle \\
&= \nabla f + \nabla g
\end{aligned}$$

QED

Directional Derivatives

Definition

Let $f(x, y) \in C^{(1,1)}$. Then, the **directional derivative** of f at point $P_0(x_0, y_0)$ in the direction of unit vector $\widehat{\mathbf{v}} \in \mathbb{R}^2$ is:

$$D_{\mathbf{v}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \widehat{\mathbf{v}}$$

Definition

Let $f(x, y, z) \in C^{(1,1,1)}$. Then, the **directional derivative** of f at point $P_0(x_0, y_0, z_0)$ in the direction of unit vector $\widehat{\mathbf{v}} \in \mathbb{R}^3$ is:

$$D_{\mathbf{v}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \widehat{\mathbf{v}}$$

REMARK: Directional Derivatives are generalizations of 1^{st}-order partials:

$$D_{\mathbf{i}} f(x, y, z) = \nabla f \cdot \widehat{\mathbf{i}} = \langle f_x, f_y, f_z \rangle \cdot \langle 1, 0, 0 \rangle = (f_x)(1) + (f_y)(0) + (f_z)(0) = f_x = \frac{\partial f}{\partial x}$$

Directional Derivatives

WEX 11-6-3: Let $f(x, y) = x^2 - y^2$ and $\mathbf{u} = \langle 3, 5 \rangle$. Compute $D_{\mathbf{u}} f(1, 2)$.

1^{st} normalize $\mathbf{u}$: $\quad \widehat{\mathbf{u}} = \dfrac{\mathbf{u}}{||\mathbf{u}||} = \dfrac{\langle 3, 5 \rangle}{\sqrt{3^2 + 5^2}} = \left\langle \dfrac{3}{\sqrt{34}}, \dfrac{5}{\sqrt{34}} \right\rangle$

$\nabla f = \langle f_x, f_y \rangle = \langle 2x, -2y \rangle$

$$\begin{aligned} D_{\mathbf{u}} f(1, 2) &= \nabla f(1, 2) \cdot \widehat{\mathbf{u}} \\ &= \langle 2(1), -2(2) \rangle \cdot \widehat{\mathbf{u}} \\ &= \langle 2, -4 \rangle \cdot \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle \\ &= (2)\left(\frac{3}{\sqrt{34}}\right) + (-4)\left(\frac{5}{\sqrt{34}}\right) \\ &= \boxed{-\frac{14}{\sqrt{34}}} \end{aligned}$$

Gradient (Steepest Ascent/Descent)

Proposition

Given surface $z = f(x,y)$ *such that* $f \in C^{(1,1)}$. *Then:*

(i) ∇f *is **normal** to a level curve that contains point* (x,y).

(ii) ∇f *points in the (compass) direction of **steepest ascent** of* f *from* (x,y).

(iii) $-\nabla f$ *points in the direction of **steepest descent** of* f *from point* (x,y).

(iv) $||\nabla f||$ *is the **maximum rate of change** of* f *at point* (x,y).

REMARK:

At point (x,y), f increases most rapidly in the direction of ∇f.
At point (x,y), f decreases most rapidly in the direction of $-\nabla f$.

PROOF:

(iv): Let unit vector $\hat{\mathbf{v}} \in \mathbb{R}^2$ point in direction of **steepest ascent**.
Let $\theta \in [0,\pi]$ be the angle between vectors $\hat{\mathbf{v}}$ & ∇f.

$$\implies D_{\mathbf{v}} f = \nabla f \cdot \hat{\mathbf{v}} = ||\nabla f|| ||\hat{\mathbf{v}}|| \cos\theta = ||\nabla f||(1)\cos\theta = ||\nabla f|| \cos\theta$$

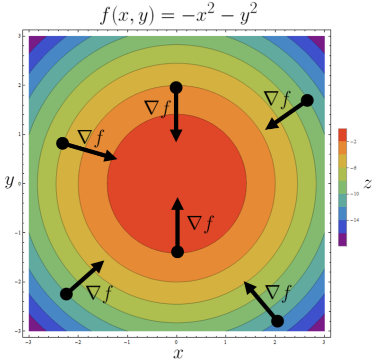
$$\implies \max(D_{\mathbf{v}} f) = \max_{\theta\in[0,\pi]} (||\nabla f|| \cos\theta) = ||\nabla f|| \max_{\theta\in[0,\pi]} (\cos\theta) = ||\nabla f||(1) = ||\nabla f||$$

(ii): $\max_{\theta\in[0,\pi]} (\cos\theta) = 1 \implies \theta = 0 \implies \hat{\mathbf{v}}$ & ∇f point in same direction.

QED

Gradient (Steepest Ascent/Descent)

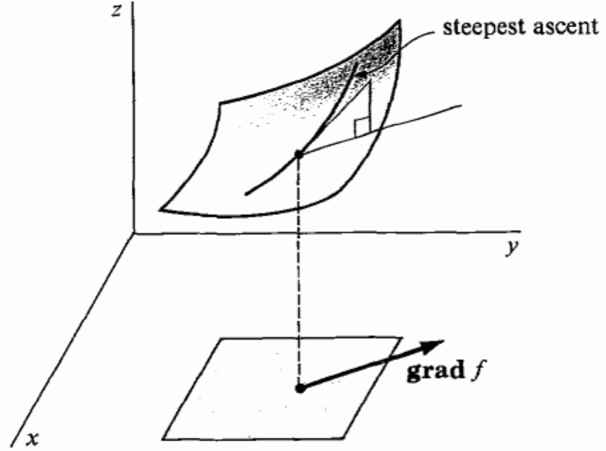
Given the **contour plot** (level curve plot) of a surface $z = f(x, y)$.



Gradient ∇f is **normal** to a level curve that contains point (x, y).
∇f points in the (compass) direction of **steepest ascent** of f from point (x, y).
$-\nabla f$ points in the direction of **steepest descent** of f from point (x, y).
$||\nabla f||$ is the **maximum rate of change** of f at point (x, y).

Gradient (Steepest Ascent)

Remember, given a surface $z = f(x, y)$:



∇f lies on the xy-plane & points in **compass direction** of **steepest ascent**.

Gradient (Steepest Ascent/Descent)

WEX 11-6-4: Given $f(x,y) = 3\tan(\pi xy)$:
In what direction is f increasing most rapidly at point $P(2,3)$?
What is the maximum rate of increase?
In what direction is f decreasing most rapidly at point $P(2,3)$?

$$\nabla f = \langle f_x, f_y \rangle = \langle 3\pi y \sec^2(\pi xy), 3\pi x \sec^2(\pi xy) \rangle$$

$$\begin{aligned} \implies \nabla f(2,3) &= \langle 3\pi(3)\sec^2[\pi(2)(3)], 3\pi(2)\sec^2[\pi(2)(3)] \rangle \\ &= \langle 9\pi \sec^2(6\pi), 6\pi \sec^2(6\pi) \rangle \\ &= \langle 9\pi \sec^2(0), 6\pi \sec^2(0) \rangle \\ &= \langle 9\pi, 6\pi \rangle \end{aligned}$$

$\therefore f$ increases most rapidly at point P in the direction of $\boxed{\langle 9\pi, 6\pi \rangle}$

$\therefore f$ decreases most rapidly at point P in the direction of $\boxed{\langle -9\pi, -6\pi \rangle}$

$$||\nabla f(2,3)|| = ||\langle 9\pi, 6\pi \rangle|| = \sqrt{(9\pi)^2 + (6\pi)^2} = \sqrt{117\pi^2} = \pi\sqrt{117}$$

$\therefore$ The maximum rate of increase from point P is $\boxed{\pi\sqrt{117}} \approx 41.8$

PART II:
TANGENT PLANES

Tangent Plane & Normal Line to a Level Surface

Proposition

*Given **level surface** $F(x, y, z) = k$, where $k \in \mathbb{R}$ and $F \in C^{(1,1,1)}$.*
Let $P_0(x_0, y_0, z_0)$ be a point on the level surface s.t. $\nabla F(x_0, y_0, z_0) \neq \vec{\mathbf{0}}$. Then:
*The equation of the **tangent plane** $\mathbb{T}$ to the level surface at point P_0 is:*

$$\mathbb{T} : F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

*The equation of the **normal line** ℓ to the level surface at point P_0 is:*

$$\ell(t) = \langle x_0 + F_x(x_0, y_0, z_0)t, y_0 + F_y(x_0, y_0, z_0)t, z_0 + F_z(x_0, y_0, z_0)t \rangle$$

s.t. $\equiv$ "such that"

WEX 11-6-5: Given $F(x,y,z) = x^2y^3z^4$:
(a) Find equation of tangent plane $\mathbb{T}$ to level surface $F(x,y,z) = 1$ at $(1,1,1)$.

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle \implies \nabla F(1,1,1) = \langle 2,3,4 \rangle$

Observe that $\nabla F(1,1,1)$ is a **normal vector** to the desired tangent plane $\mathbb{T}$.
Hence:

$2x + 3y + 4z + D = 0 \implies 2(1) + 3(1) + 4(1) + D = 0 \implies D = -9$

$\therefore$ Equation of tangent plane is $\mathbb{T}$: $\boxed{2x + 3y + 4z - 9 = 0}$

(b) Find equation of normal line ℓ to level surface $F(x,y,z) = 1$ at point $(1,1,1)$.

The normal line $\ell \,||$ gradient $\nabla F(1,1,1) = \langle 2,3,4 \rangle$ and contains point $(1,1,1)$.

$\therefore$ Equation of normal line to level surface is $\ell(t) = \boxed{\langle 1+2t, 1+3t, 1+4t \rangle}$

Tangent Plane & Normal Line to Surface $z = f(x, y)$

WEX 11-6-6: Given surface $S : z = -x^2 - y^2$:
(a) Find equation of tangent plane $\mathbb{T}$ to surface S at point $(1, 1, -2)$.

1^{st} write surface S as a **level surface**:
$z = -x^2 - y^2 \implies F(x, y, z) := z + x^2 + y^2 = 0$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 1 \rangle \implies \nabla F(1, 1, -2) = \langle 2, 2, 1 \rangle$

Observe that $\nabla F(1, 1, -2)$ is a **normal vector** to the desired tangent plane $\mathbb{T}$.
Hence:

$2x + 2y + z + D = 0 \implies 2(1) + 2(1) + (-2) + D = 0 \implies D = -2$

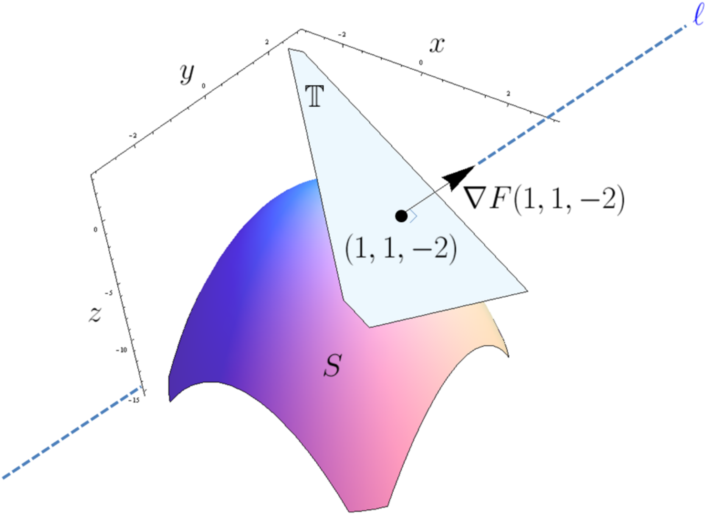
$\therefore$ Equation of tangent plane is $\mathbb{T}$: $\boxed{2x + 2y + z - 2 = 0}$

(b) Find equation of normal line ℓ to surface S at point $(1, 1, -2)$.

The normal line $\ell \,||$ gradient $\nabla F(1, 1, -2) = \langle 2, 2, 1 \rangle$ and contains pt $(1, 1, -2)$.

$\therefore$ Equation of normal line to level surface is $\ell(t) = \boxed{\langle 1 + 2t, 1 + 2t, -2 + t \rangle}$

Tangent Plane $\mathbb{T}$ & Normal Line ℓ to Surface S



$$S : z = -x^2 - y^2$$

Fin.