Functions of Two Variables: Extrema

Calculus III

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TTU

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PART I:
RELATIVE EXTREMA OF MULTIVARIABLE FUNCTIONS

Open & Closed Sets in $\mathbb{R}$ and $\mathbb{R}^2$

Definition

A set $S \subseteq \mathbb{R}$ is **open** if S contains none of its boundary.

A set $S \subseteq \mathbb{R}$ is **closed** if S contains all of its boundary.

REMARK: $\mathbb{R}$ and $\emptyset$ (empty set) are **both open and closed**.

Definition

A set $S \subseteq \mathbb{R}^2$ is **open** if S contains none of its boundary.

A set $S \subseteq \mathbb{R}^2$ is **closed** if S contains all of its boundary.

REMARK: $\mathbb{R}^2$ and $\emptyset$ (empty set) are **both open and closed**.

WEX 11-7-1: Is interval $(-2, 6)$ open, closed, or neither?

Determine the **boundary** of $(-2, 6)$: Bdy$\Big[(-2, 6)\Big] = \{-2, 6\}$

Observe that $-2 \notin (-2, 6)$ and $6 \notin (-2, 6) \implies \{-2, 6\} \not\subseteq (-2, 6)$

$\therefore$ The interval contains **none** of its boundary $\implies (-2, 6)$ is $\boxed{\text{open}}$

WEX 11-7-2: Is interval $[4, 9]$ open, closed, or neither?

Determine the **boundary** of $[4, 9]$: Bdy$\Big[[4, 9]\Big] = \{4, 9\}$

Observe that $4 \in [4, 9]$ and $9 \in [4, 9] \implies \{4, 9\} \subseteq [4, 9]$

$\therefore$ The interval contains **all** of its boundary $\implies [4, 9]$ is $\boxed{\text{closed}}$

WEX 11-7-3: Is interval $[5, 7)$ open, closed, or neither?

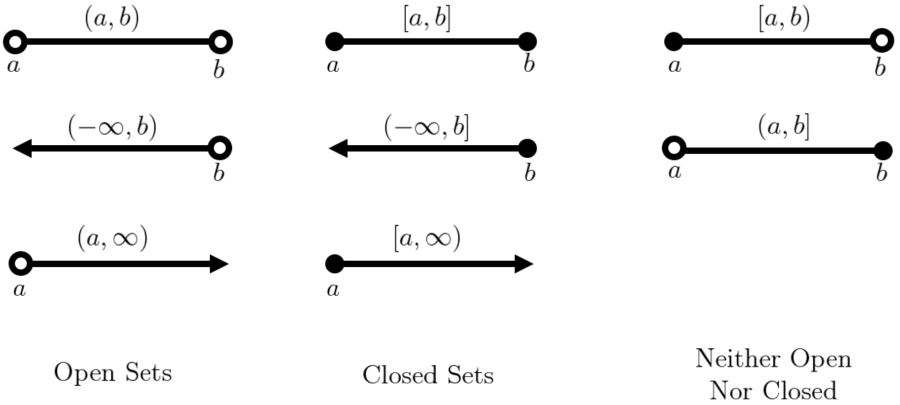
Determine the **boundary** of $[5, 7)$: Bdy$\Big[[5, 7)\Big] = \{5, 7\}$

Observe that $5 \in [5, 7)$ and $7 \notin [5, 7)$

$\therefore$ The interval contains **only part** of its boundary

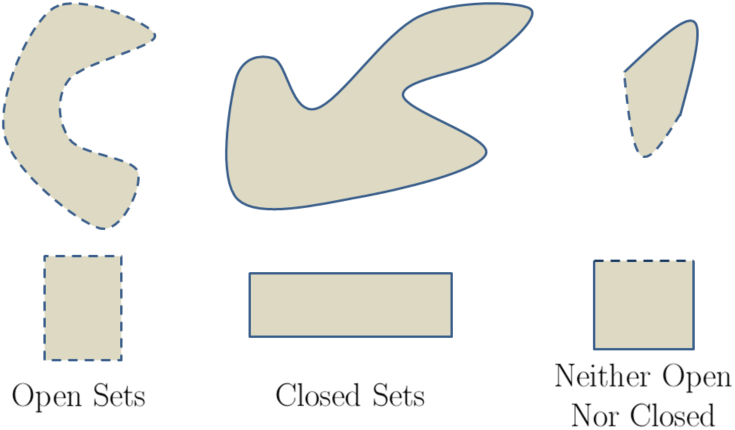
$\implies [5, 7)$ is $\boxed{\text{neither open nor closed}}$

Open & Closed Sets in $\mathbb{R}$



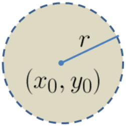
Hollow circle(s) on the boundary indicate they are **not** part of the set.

Open & Closed Sets in $\mathbb{R}^2$

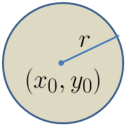


Dashed portions of the boundary indicate they are **not** part of the set.

Open & Closed Disks in $\mathbb{R}^2$



Open Disk
centered at (x_0, y_0)
with radius r



Closed Disk
centered at (x_0, y_0)
with radius r

Definition

The **open disk** centered at (x_0, y_0) with radius $r > 0$ is defined by:

$$\mathbb{D}(x_0, y_0; r) := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\right\}$$

Definition

The **closed disk** centered at (x_0, y_0) with radius $r > 0$ is defined by:

$$\overline{\mathbb{D}}(x_0, y_0; r) := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\right\}$$

Relative Extrema & Saddle Points (First Principles)

Definition

Let $f(x,y)$ be defined on an open set $S \subseteq \mathbb{R}^2$ such that $(x_0,y_0) \in S$. Then:

(x_0,y_0) is a **relative maximum** if $f(x,y) \le f(x_0,y_0) \quad \forall (x,y) \in \mathbb{D}(x_0,y_0;r)$.
(x_0,y_0) is a **relative minimum** if $f(x,y) \ge f(x_0,y_0) \quad \forall (x,y) \in \mathbb{D}(x_0,y_0;r)$.
(x_0,y_0) is a **saddle point** if open disk $\mathbb{D}(x_0,y_0;r)$ contains points s.t. $f(x,y) > f(x_0,y_0)$ as well as points s.t. $f(x,y) < f(x_0,y_0)$.

$\forall \equiv$ ”for all” or ”for every” or ”for each”

Definition

A **relative extremum** is either a relative max or a relative min.

Unfortunately, these ”first principles” definitions of relative extrema & saddle points are often too tedious to use.

What follows are simpler definitions using **partial derivatives**.

Critical Points

Recall from Calculus I:

Definition

Let $f(x)$ be defined on an open set $S \subseteq \mathbb{R}$ such that $x_0 \in S$.
Then x_0 is a **critical number** of f if either one of the following is true:

(i) $f'(x_0) = 0$
(ii) $f'(x_0)$ DNE

Here's the corresponding terminology for a function of two variables:

Definition

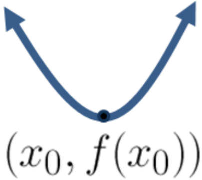
Let $f(x, y)$ be defined on an open set $S \subseteq \mathbb{R}^2$ such that $(x_0, y_0) \in S$.
Then (x_0, y_0) is a **critical point (CP)** of f is either one of the following is true:

(i) $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
(ii) At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ DNE

DNE $\equiv$ "**D**oes **N**ot **E**xist"

Function of One Variable (2^{nd}-Derivative Test)

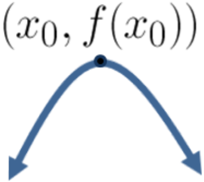
Recall from Calculus I the 2^{nd}-Derivative Test:



Relative Min at x_0

$f'(x_0) = 0$

$f''(x_0) > 0$



Relative Max at x_0

$f'(x_0) = 0$

$f''(x_0) < 0$

2^{nd}-Derivative Test is inconclusive if $f''(x_0) = 0$.
Further analysis is necessary to determine the nature of f at x_0.
Build & interpret the **slope & concavity tables**.

Function of Two Variables (Mixed 2^{nd}-Order Partials)

Theorem

(Sufficient Condition for Equality of Mixed Partials)

Let $f(x,y) \in C^{(2,2)}$.

Then $f_{xy} = f_{yx}$

PROOF: Take **Advanced Calculus**.

Examples of functions which are $C^{(2,2)}$ everywhere:

- Polynomials, Sines, Cosines, Exponentials, ArcTangents, ArcCotangents

REMARK: The arguments must be defined everywhere!

e.g. $\sin(xy) \in C^{(2,2)}(\mathbb{R}^2)$ and $e^{x^2y^5} \in C^{(2,2)}(\mathbb{R}^2)$
but $\sin(\sqrt{xy}) \notin C^{(2,2)}(\mathbb{R}^2)$ and $e^{1/xy} \notin C^{(2,2)}(\mathbb{R}^2)$

WARNING: Some functions are defined everywhere, but not $C^{(2,2)}(\mathbb{R}^2)$:

- **Odd roots**: $\sqrt[3]{xy}, \sqrt[5]{xy}, \ldots$
- **Absolute values**: $|xy|, |x+y|, \ldots$

Function of Two Variables (2^{nd}-Order Partials Test)

Theorem

Let $f(x,y) \in C^{(2,2)}\left(\mathbb{D}(x_0,y_0;r)\right)$ *s.t.* f *has a critical point at* (x_0,y_0).

Form the ***discriminant*** *of* f*:* $\Delta(x,y) :=$ *det* $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - (f_{xy})^2$

Then:

(x_0,y_0) *is a* ***relative max*** *(AKA* ***local max****) if*
$\left(\Delta(x_0,y_0) > 0 \text{ and } f_{xx}(x_0,y_0) < 0\right)$ *OR* $\left(\Delta(x_0,y_0) > 0 \text{ and } f_{yy}(x_0,y_0) < 0\right)$

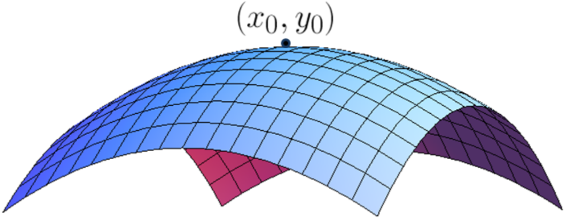
(x_0,y_0) *is a* ***relative min*** *(AKA* ***local min****) if*
$\left(\Delta(x_0,y_0) > 0 \text{ and } f_{xx}(x_0,y_0) > 0\right)$ *OR* $\left(\Delta(x_0,y_0) > 0 \text{ and } f_{yy}(x_0,y_0) > 0\right)$

(x_0,y_0) *is a* ***saddle point*** *if* $\Delta(x_0,y_0) < 0$

The test is ***inconclusive*** *if* $\Delta(x_0,y_0) = 0$.

PROOF: Requires use of **Taylor Series in Two Variables**, which is covered in **Advanced Calculus** and **Numerical Analysis**.

2^{nd}-Order Partials Test (Relative Max)

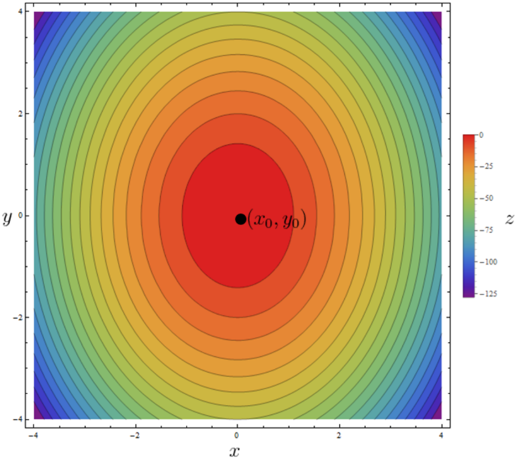


Relative Max occurs at point (x_0, y_0) if:

$\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$

— OR —

$\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$



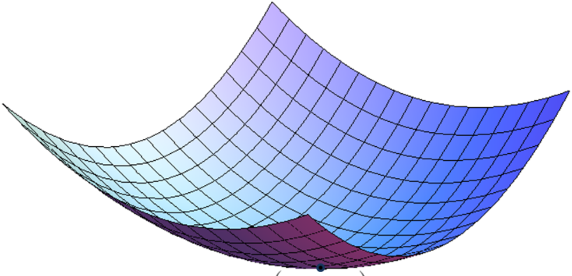
Relative Max occurs at point (x_0, y_0) if:

$\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$

— OR —

$\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$

2^{nd}-Order Partials Test (Relative Min)



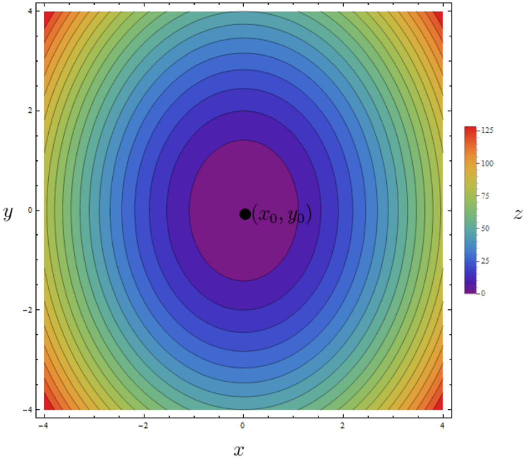
Relative Min occurs at point (x_0, y_0) if:

$\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$

— OR —

$\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$

2^{nd}-Order Partials Test (Relative Min – Contour Plot)



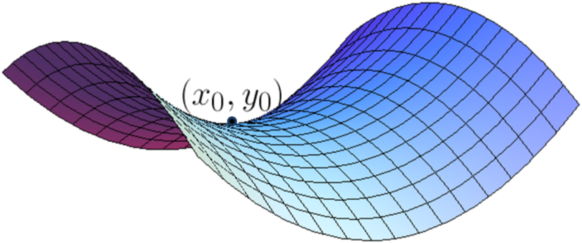
Relative Min occurs at point (x_0, y_0) if:

$\Delta(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$

— OR —

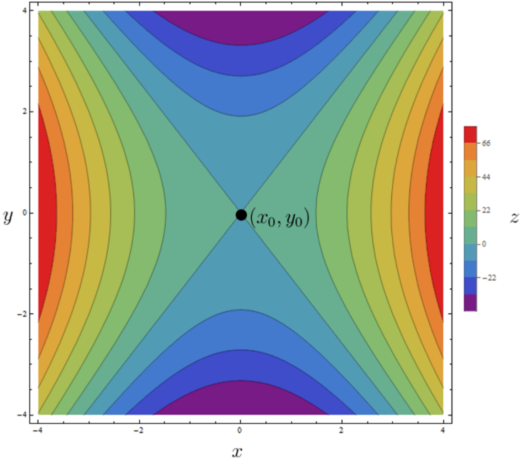
$\Delta(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$

2^{nd}-Order Partials Test (Saddle Point)



Saddle Point occurs at point (x_0, y_0) if:
$\Delta(x_0, y_0) < 0$

2^{nd}-Order Partials Test (Saddle Point – Contour Plot)



Saddle Point occurs at point (x_0, y_0) if:

$$\Delta(x_0, y_0) < 0$$

2^{nd}-Order Partials Test (Inconclusive)

If $\Delta(x_0, y_0) = 0$, then the 2^{nd}-Order Partials Test is **inconclusive**.
Further analysis is necessary to determine the nature of f at (x_0, y_0).

Apply the ”first principles” definitions of rel max, rel min, and saddle point.

2^{nd}-Order Partials Test (Example)

WEX 11-7-4: Let $f(x,y) = x^2 - xy - y^3$. Find & classify all CP's of f.

$$\begin{cases} f_x = 2x - y \overset{set}{=} 0 \\ f_y = -x - 3y^2 \overset{set}{=} 0 \end{cases} \implies \begin{cases} y = 2x \\ x + 3y^2 = 0 \end{cases}$$

$$\implies x + 3(2x)^2 = 0 \implies x + 12x^2 = 0 \implies x(1+12x) = 0$$

$$\implies \left(x = 0 \implies y = 2(0) = 0\right) \text{ or } \left(x = -\tfrac{1}{12} \implies y = 2\left(-\tfrac{1}{12}\right) = -\tfrac{1}{6}\right)$$

$\therefore$ The critical points (CP's) of f are $\boxed{(0,0), \left(-\frac{1}{12}, -\frac{1}{6}\right)}$

$$f_{xx} = 2, \quad f_{yy} = -6y, \quad f_{xy} = -1$$

$$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = (2)(-6y) - (-1)^2 = -12y - 1$$

CP	$(0,0)$	$\left(-\frac{1}{12}, -\frac{1}{6}\right)$
Δ	$-$	$+$
f_{xx}	DC	$+$
f_{yy}	DC	$+$
Type	Saddle Point	Relative Minimum

DC $\equiv$ "Don't Care"

PART II:

ABSOLUTE EXTREMA OF MULTIVARIABLE FUNCTIONS

Bounded Sets in $\mathbb{R}$

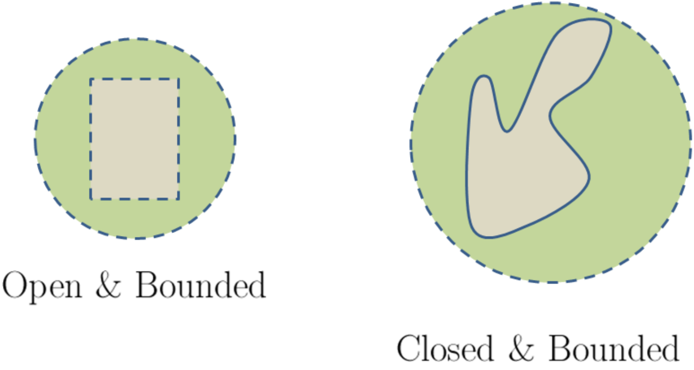
Definition

A set $S \subset \mathbb{R}$ is **bounded** if S is contained in an open interval.

$\Big($Here, $-\infty < a < b < \infty\Big)$

- $(a,b) \subset (a-1,b+1) \implies (a,b)$ is open & bounded
- $[a,b] \subset (a-1,b+1) \implies [a,b]$ is closed & bounded
- $\mathbb{R}, (-\infty,b), (-\infty,b], (a,\infty), [a,\infty)$ are all **unbounded**

Bounded Sets in $\mathbb{R}^2$



Definition

A set $S \subset \mathbb{R}^2$ is **bounded** if S is contained in an open disk.

REMARK: $\mathbb{R}^2$ and each **quadrant** of the xy-plane are all **unbounded**

Absolute Extrema ("First Principles" Definitions)

Definition

Given function $f(x,y)$:
(x_M,y_M) is an **absolute maximum** of f if $f(x_M,y_M) \geq f(x,y) \quad \forall (x,y) \in$ Dom(f)
(x_m,y_m) is an **absolute minimum** of f if $f(x_m,y_m) \leq f(x,y) \quad \forall (x,y) \in$ Dom(f)

If (x_M,y_M) is an abs max of f, then $f(x_M,y_M)$ is the **absolute max value** of f.
If (x_m,y_m) is an abs min of f, then $f(x_m,y_m)$ is the **absolute min value** of f.

The **extreme values** of f are the **abs max value & abs min value** of f.

The absolute max is also known as the **global max**. Similarly for the abs min.

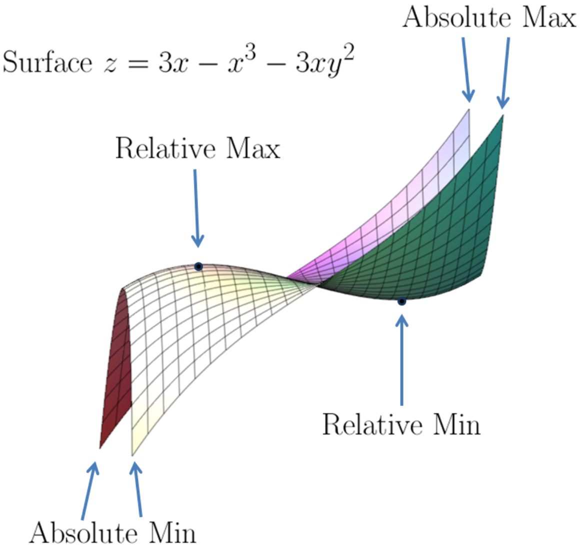
Theorem

(Extreme Value Theorem or E-V-T)

*Let $f(x,y) \in C(S)$ where set $S \subset \mathbb{R}^2$ is **closed & bounded**.*
Then f attains extreme values over the set S.

PROOF: Take **Advanced Calculus**.

Absolute Extrema (Surface Plot)



Absolute Extrema (Procedure)

Proposition

*Let $f(x,y) \in C(S)$ where set $S \subset \mathbb{R}^2$ is **closed & bounded**.*
Then to find the absolute extrema of f over S, follow this procedure:

** Find all critical points (CP's) of f.*

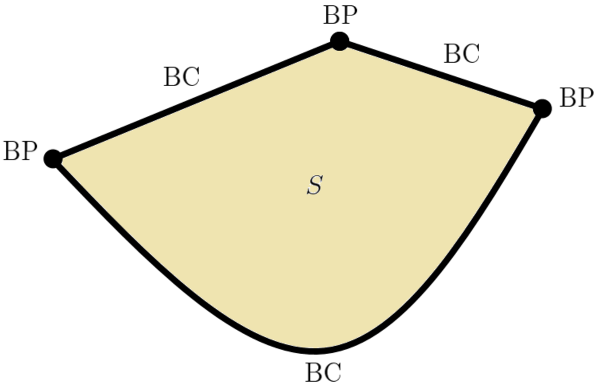
** Sketch & label all **boundary curves (BC's) & boundary points (BP's)** of S.*
*A **boundary point** is the **intersection** of two boundary curves.*
** Discard any critical points (CP's) that are <u>not</u> in S.*

** Find all points on the **boundary** of S where absolute extrema can occur (called **boundary critical points (BCP's)**).*
To do this, find the absolute extrema on a function of <u>one variable</u> by plugging in one of the BC's of S. Repeat this for each BC.

** Build a **table** by computing f for each CP, BP, and BCP.*
** The abs max value of f is the largest of all computed values in table.*
** The abs min value of f is the smallest of all computed values in table.*

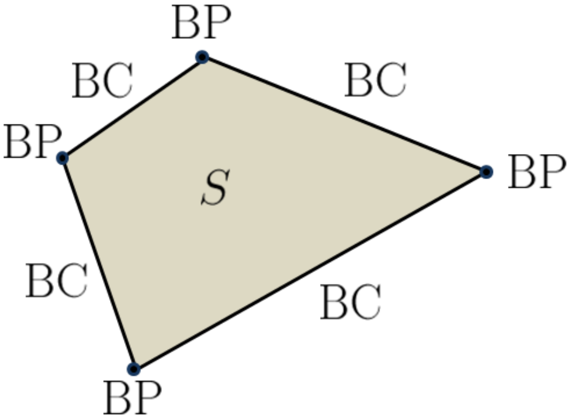
REMARK: Sometimes it's best to **parameterize** the BC.

Boundary Curves (BC's) & Boundary Points (BP's)



A **boundary point (BP)** is the **intersection** of two boundary curves (BC's).

Boundary Curves (BC's) & Boundary Points (BP's)



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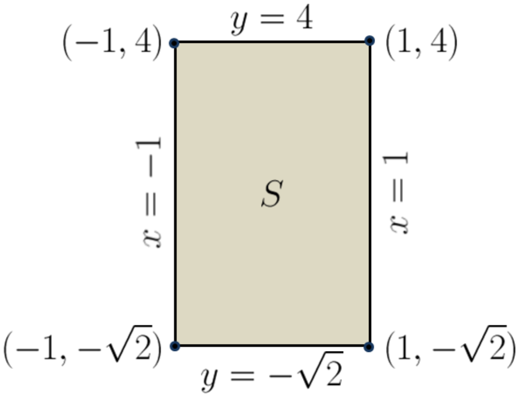
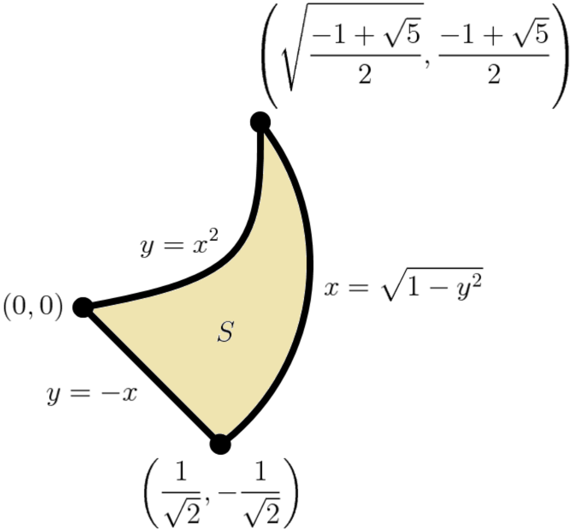
Boundary Curves (BC's) & Boundary Points (BP's)



$$
\begin{aligned}
S &= \left\{(x, y) \in \mathbb{R}^2 : |x| \leq 1, -\sqrt{2} \leq y \leq 4\right\} \\
&= \left\{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -\sqrt{2} \leq y \leq 4\right\}
\end{aligned}
$$

Boundary Curves (BC's) & Boundary Points (BP's)



Fin.