Constrained Optimization: Lagrange Multipliers

Calculus III

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Lagrange's Theorem

Theorem

(Lagrange's Theorem in Two Variables)
Let $f, g \in C^{(1,1)}$ *s.t.* $f(x_0, y_0)$ *is an* ***extreme value*** *when restricted to the* ***constraint*** $g(x, y) = k$*, where* $k \in \mathbb{R}$*.*

If $\nabla g(x_0, y_0) \neq \vec{\mathbf{0}}$*, then there exists a* ***Lagrange multiplier*** $\lambda \in \mathbb{R}$ *s.t.*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Theorem

(Lagrange's Theorem in Three Variables)
Let $f, g \in C^{(1,1,1)}$ *s.t.* $f(x_0, y_0, z_0)$ *is an* ***extreme value*** *when restricted to the* ***constraint*** $g(x, y, z) = k$*, where* $k \in \mathbb{R}$*.*

If $\nabla g(x_0, y_0, z_0) \neq \vec{\mathbf{0}}$*, then there exists a* ***Lagrange multiplier*** $\lambda \in \mathbb{R}$ *s.t.*

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

PROOF: See the textbook – it's somewhat lengthy.

Lagrange Multipliers (Procedure)

Proposition

(Method of Lagrange in Two Variables)
Let $f, g \in C^{(1,1)}$ *s.t.* $f(x,y)$ *has an extremum subject to constraint* $g(x,y) = k$*, where* $k \in \mathbb{R}$*.*
Then to find the extreme value(s) of f*: Solve system:* $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$

STEP 1: *Solve for* x, y*:* $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = k \end{cases}$

The solutions (x,y) *are called* ***constrained critical points (CCP's)*** *of* f*.*

STEP 2: *Build a table computing* f *for each constrained critical point (CCP) If there's only one CCP, pick any other (simple) point on the constraint to compare with.*

STEP 3: *Compare values in the table to determine the extreme value(s) of* f*.*

Lagrange Multipliers (Procedure)

Proposition

(Method of Lagrange in Three Variables)
Let $f, g \in C^{(1,1,1)}$ *s.t.* $f(x,y,z)$ *has an extremum subject to constraint* $g(x,y,z) = k$, *where* $k \in \mathbb{R}$.

Then to find the extreme value(s) of f*: Solve system:* $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = k \end{cases}$

STEP 1: *Solve for* x, y, z*:* $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = k \end{cases}$

The solutions (x,y,z) *are called* ***constrained critical points (CCP's)*** *of* f.

STEP 2: *Build a table computing* f *for each constrained critical point (CCP)*
If there's only one CCP, pick any other (simple) point on the constraint to compare with.

STEP 3: *Compare values in the table to determine the extreme value(s) of* f.

Lagrange Multipliers (Example)

WEX 11-8-1: Find the extreme values of $f(x,y) = xy$ s.t. constraint $x^2 + y^2 = 1$.

Let $g(x,y) = x^2 + y^2$. Then, $\nabla f = \langle y, x\rangle$ and $\nabla g = \langle 2x, 2y\rangle$

Solve nonlinear system $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases} \iff \begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x,y) = 1 \end{cases} \iff \begin{cases} y = 2x\lambda \\ x = 2y\lambda \\ x^2 + y^2 = 1 \end{cases}$

Observe that the nonlinear system forces $x \neq 0$ and $y \neq 0 \implies \lambda \neq 0$

$\begin{cases} y = 2x\lambda \\ x = 2y\lambda \end{cases} \implies$ (Divide top eqn by bottom eqn) $\implies \frac{y}{x} = \frac{x}{y} \implies x^2 = y^2$

Then, $x^2 + y^2 = 1 \implies x^2 + x^2 = 1 \implies 2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}} \implies y = \pm\frac{1}{\sqrt{2}}$

$\therefore$ The CCP's of f are: $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Build table computing f at each CCP and then compare the values:

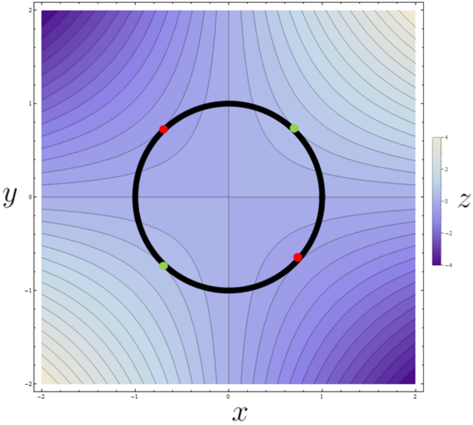
(x,y)	$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$f(x,y)$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	Abs Max	**Abs Min**	**Abs Min**	**Abs Max**

s.t. $\equiv$ "subject to"

Lagrange Multipliers (Geometric Interpretation)

Optimize $f(x, y) = xy$ subject to constraint $g(x, y) := x^2 + y^2 = 1$

contour plot of surface $z = xy$ along with curve $x^2 + y^2 = 1$

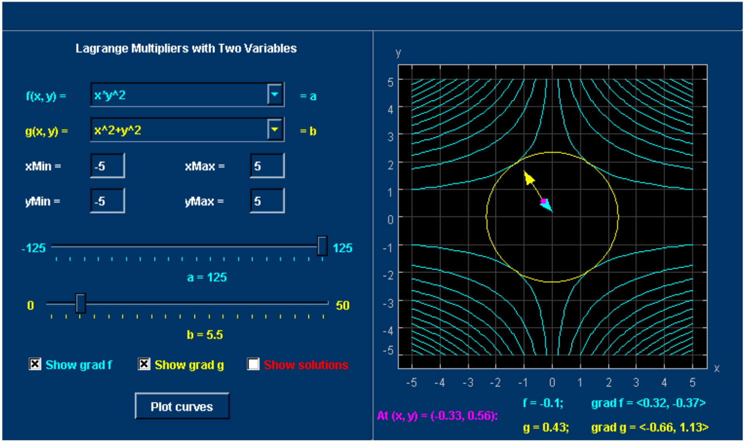


At an absolute min (**red point**) or absolute max (**green point**):

$\nabla f = \lambda \nabla g \iff$ the gradients of f and g are **parallel**

and the curve $g(x, y) = 1$ is **tangent** to a level curve of $f(x, y)$.

(DEMO) LAGRANGE MULTIPLIERS (Click below):



Fin.