Double Integrals: Rectangular Coordinates

Calculus III

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TTU

13 October 2014

PART I:

SETUP & COMPUTATION OF DOUBLE INTEGRALS

Indefinite Integral Rules (from Calculus I)

Here, $C \in \mathbb{R}$ is called the **constant of integration**.
Also, $k \in \mathbb{R}$.

Zero Rule: $$\int 0 \, dx = C$$

Constant Rule: $$\int k \, dx = kx + C$$

Constant Multiple Rule: $$\int kf(x) \, dx = k \int f(x) \, dx$$

Sum/Diff Rule: $$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

Power Rule: $$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \qquad \Big(\text{provided } n \in \mathbb{R} \setminus \{-1\}\Big)$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \qquad \Big(\text{provided } a \in \mathbb{R}_+ \setminus \{1\}\Big)$$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

Indefinite Integral Rules (from Calculus I)

Here, $C \in \mathbb{R}$ is called the **constant of integration**.

- $\displaystyle\int \sin x \, dx = -\cos x + C$
- $\displaystyle\int \cos x \, dx = \sin x + C$
- $\displaystyle\int \sec^2 x \, dx = \tan x + C$
- $\displaystyle\int \sec x \tan x \, dx = \sec x + C$
- $\displaystyle\int \csc^2 x \, dx = -\cot x + C$
- $\displaystyle\int \csc x \cot x \, dx = -\csc x + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$
- $\displaystyle\int \frac{1}{1+x^2} \, dx = \arctan x + C$
- $\displaystyle\int \frac{1}{|x|\sqrt{x^2-1}} \, dx = \text{arcsec}\, x + C$

WORKED EXAMPLE: Evaluate $\int xe^{x^2}\,dx$.

CV: Let $u = x^2$, then $du = 2x\,dx \implies x\,dx = \frac{1}{2}\,du$

$$\implies \int xe^{x^2}\,dx \stackrel{CV}{=} \int e^u \left(\frac{1}{2}\,du\right) = \frac{1}{2}e^u + C \stackrel{CV}{=} \boxed{\frac{1}{2}e^{x^2} + C}$$

WORKED EXAMPLE: Evaluate $\int_{-2}^{3} xe^{x^2}\,dx$.

CV: Let $u = x^2$, then $du = 2x\,dx \implies x\,dx = \frac{1}{2}\,du$

and $u(-2) = (-2)^2 = 4$ and $u(3) = (3)^2 = 9$

$$\implies \int_{-2}^{3} xe^{x^2}\,dx \stackrel{CV}{=} \int_{4}^{9} e^u \left(\frac{1}{2}\,du\right) = \left[\frac{1}{2}e^u\right]_{u=4}^{u=9} \stackrel{FTC}{=} \boxed{\frac{1}{2}\left(e^9 - e^4\right)}$$

Advanced Integration Techniques (from Calculus II)

- Integration by Parts (IBP)

$$\int xe^x\,dx \overset{IBP}{=} xe^x - \int e^x\,dx = \boxed{xe^x - e^x + C}$$

- Partial Fraction Decomposition (PFD)

$$\int \frac{1}{x(x+2)}\,dx \overset{PFD}{=} \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+2}\,dx = \boxed{\frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + C}$$

- Trig Integrals

$$\int \sin^6 x\cos^5 x\,dx,\ \int \cos^4\omega\,d\omega,\ \int \tan^3 t\,dt,\ \int \tan^4 x\sec^6 x\,dx,$$
$$\int \sin(13x)\cos(7x)\,dx,\ \int \cos(-13\theta)\cos(7\theta)\,d\theta,\ \int \sin(13t)\sin(-7t)\,dt,\ \ldots$$

- Trig Substitution

$$\int \frac{1}{\sqrt{1-x^2}}\,dx,\ \int x^3\sqrt{1-x^2}\,dx,\ \int \sqrt{1-x^2}\,dx,\ \int \frac{x^3}{\sqrt{4-x^2}}\,dx,$$
$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx,\ \int \frac{\sqrt{4-x^2}}{x^2}\,dx,\ \int \frac{1}{\sqrt{2x^2-8x+6}}\,dx,\ \ldots$$

Integration by Parts (Purpose)

Here, assume that u, v are functions of x: $\quad u \equiv u(x), \quad v \equiv v(x)$

$$\text{IBP:} \quad \underbrace{\int u \, dv}_{\text{Hard}} = uv - \underbrace{\int v \, du}_{\text{Easier}}$$

$$\text{IBP:} \quad \underbrace{\int_a^b u \, dv}_{\text{Hard}} = \Big[uv\Big]_{x=a}^{x=b} - \underbrace{\int_a^b v \, du}_{\text{Easier}}$$

But the question is: How to choose function u & differential dv???

IBP: Choosing the right function u (LIPTE Heuristic)

LIPTE Heuristic:

Whichever function type comes first in the following list choose as u:

Letter	Function Type	Example Functions
L	Logarithms	$\ln x, \log y, \log_8 t, \ldots$
I	Inverse Trig	$\arcsin x, \arctan y, \text{arcsec}\, t, \ldots$
P	Polynomials	$x, y^2, 5t^3, \ldots$
T	Trig Functions	$\sin x, \tan\theta, \sec\omega, \ldots$
E	Exponentials	$e^x, 2^y, \left(\frac{1}{3}\right)^t, (-4)^x, \ldots$

There's no **R** in LIPTE, so **rational fcns** & **roots** are disqualified to be u.

Once u is chosen, the remainder of the integrand must be dv.

If integrand involves **only rational fcns and/or roots**, IBP is no good:

$$\int \frac{dx}{(x-1)(x+7)}, \quad \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}, \quad \int \frac{1+\sqrt{x}}{x^2+1}\,dx, \quad \text{etc...}$$

Definition

A **nonelementary integral** is an integral whose antiderivative cannot be expressed in a finite closed form.

In other words, the antiderivative is an **infinite series**.

Here's a small list of nonelementary integrals (there are many, many more):

$$\int e^{x^2}\,dx \qquad \int \frac{e^x}{x}\,dx \qquad \int \sqrt{x}e^{-x}\,dx$$

$$\int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx \qquad \int e^{\cos x}\,dx$$

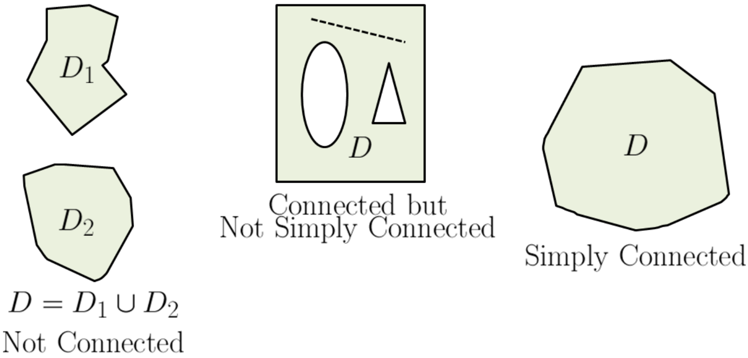
$$\int \sqrt{1+x^4}\,dx \qquad \int \ln(\ln x)\,dx \qquad \int \frac{x}{e^x-1}\,dx$$

$$\int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx \qquad \int \sin(\sin x)\,dx$$

$$\int x^x\,dx \qquad \int \frac{1}{x^x}\,dx \qquad \int \arctan(\ln x)\,dx$$

REMARK: Here in Calculus III, avoid nonelementary integrals at all costs!

Connectedness & Regions in $\mathbb{R}^2$



Definition

A **simply connected set** in $\mathbb{R}^2$ is a connected set with no holes or cuts.
A **region** is a **simply connected set** in $\mathbb{R}^2$.

(Single) Integrals (Definition & Interpretation)

Definition

(Riemann Sum Definition of an Integral)
Let $f \in C[a,b]$ where $[a,b]$ is a **closed interval** s.t. $-\infty < a < b < \infty$. Then:

$$\int_a^b f(x)\,dx := \lim_{N\to\infty} \sum_{k=1}^{N} f\left(x_k^*\right)\Delta x$$

Proposition

(The Integral as an Area)
Let $f \in C[a,b]$ s.t. $f(x) \geq 0 \ \ \forall x \in [a,b]$.
Let R be the region bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ & $x = b$. Then

$$\textit{Area}(R) = \int_a^b f(x)\,dx$$

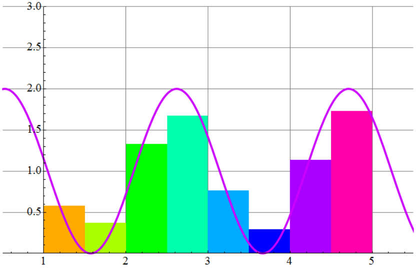
(DEMO) RIEMANN SUM DEFINITION OF AN INTEGRAL (Click below):

subdivision multiplier n 1

rectangle heights mean

RIEMANN SUMS by with enhancements by Josh Engwer (2013) [CC-BY-SA]

subdivisions of a unit interval	2	integral of $\sin(3x)+1$ from 1 to 5	3.92323
error	−0.0149645	Riemann sum	3.9382



Double Integrals (Definition)

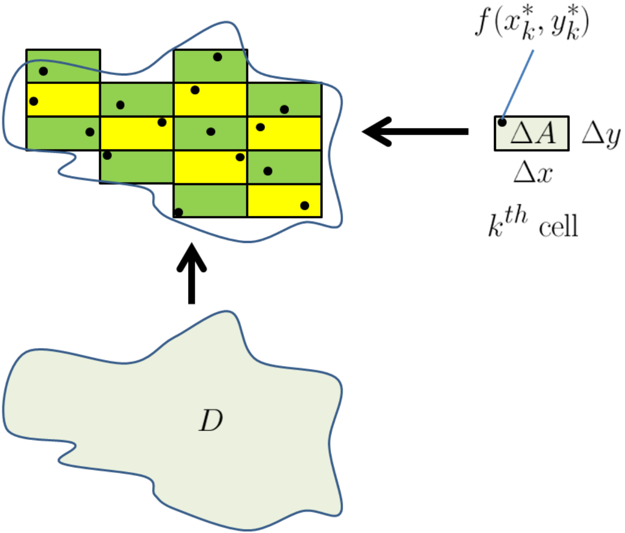
Definition

(Riemann Sum Definition of a Double Integral)
Let $f(x,y) \in C(D)$ where $D \subset \mathbb{R}^2$ is a closed & bounded region on the xy-plane. Then:

$$\iint_D f\, dA \equiv \iint_D f(x,y)\, dA := \lim_{N\to\infty} \sum_{k=1}^{N} f\left(x_k^*, y_k^*\right) \Delta A$$

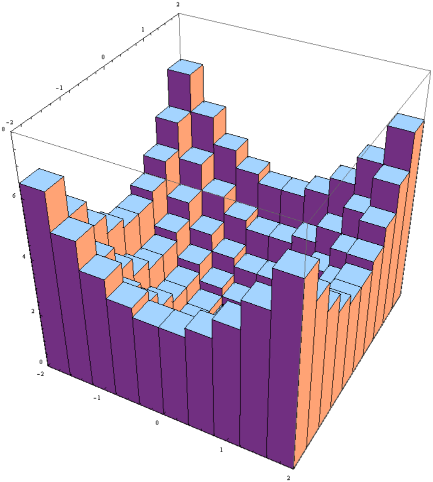
REMARK: Region D is also known as the **region of integration**.

Subdivision of closed & bounded region D into N cells



Riemann Sum Definition of a Double Integral (Demo)

(DEMO) RIEMANN SUM DEFN OF A DOUBLE INTEGRAL (Click below):



Double Integrals (Geometric Interpretation)

Proposition

(The Double Integral as a Volume)
Let $D \subset \mathbb{R}^2$ be a closed & bounded ***region*** *on the xy-plane.*
Let $f \in C(D)$ s.t. $f(x,y) \geq 0 \ \ \forall (x,y) \in D$.
Let $E \subset \mathbb{R}^3$ be the ***solid*** *in xyz-space bounded below by the xy-plane and bounded above by the surface $z = f(x,y)$. Then*

$$\textit{Volume}(E) = \iint_D f \, dA$$

Proposition

(The Double Integral as an Area)
Let $D \subset \mathbb{R}^2$ be a closed & bounded region on the xy-plane. Then

$$\textit{Area}(D) = \iint_D dA$$

Double Integrals (Properties)

Let set D be a closed & bounded region in $\mathbb{R}^2$.
Let functions $f(x,y)$ & $g(x,y)$ be defined on D.
Let $k \in \mathbb{R}$.

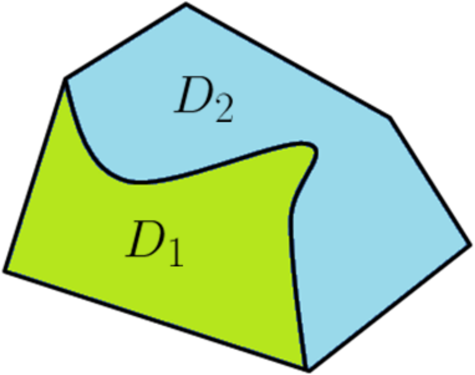
Constant Multiple Rule: $\displaystyle\iint_D kf\ dA = k\iint_D f\ dA$

Sum/Difference Rule: $\displaystyle\iint_D \Big[f \pm g\Big]\ dA = \iint_D f\ dA \pm \iint_D g\ dA$

Nonnegativity Rule: $\displaystyle f(x,y) \geq 0\ \ \forall (x,y) \in D \implies \iint_D f\ dA \geq 0$

Dominance Rule: $\displaystyle f(x,y) \leq g(x,y)\ \ \forall (x,y) \in D \implies \iint_D f\ dA \leq \iint_D g\ dA$

Double Integrals (Region Additivity Property)



Region Additivity Rule: $D = D_1 \cup D_2 \implies \iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$

Iterated Integrals (Definition)

Using the Riemann Sum Definition to compute $\iint_D f\ dA$ is too tedious & hard!

Instead write the double integral as an **iterated integral**:

$$\iint_D f\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy \right]\ dx$$

OR

$$\iint_D f\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx \right]\ dy$$

To compute an iterated integral, compute the "inner integral" first, holding the other variable constant, then compute the "outer integral."

Notice that the bounds of the "inner integral" may be functions of one variable, but the bounds of the "outer integral" are always constants.

The advantage of using iterated integrals is that all the integration rules & techniques from Calculus I & II are available.

Iterated Integrals (Examples)

WEX 12-2-1: Evaluate $I = \int_0^1 \int_2^4 xy^2 \, dx \, dy$

$$I = \int_0^1 \left[\int_2^4 xy^2 \, dx \right] dy = \Big(\text{Treat } y \text{ as constant} \Big) = \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_{x=2}^{x=4} dy$$

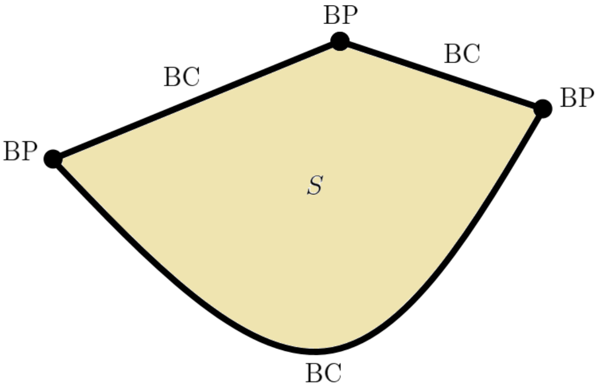
$$\overset{FTC}{=} \int_0^1 \left[\frac{1}{2}(4)^2 y^2 - \frac{1}{2}(2)^2 y^2 \right] dy = \int_0^1 6y^2 \, dy = \left[2y^3 \right]_{y=0}^{y=1} \overset{FTC}{=} 2(1)^3 - 2(0)^3 = \boxed{2}$$

WEX 12-2-2: Compute $I = \int_0^1 \int_{x^2}^{x^3} 3xy^2 \, dy \, dx$

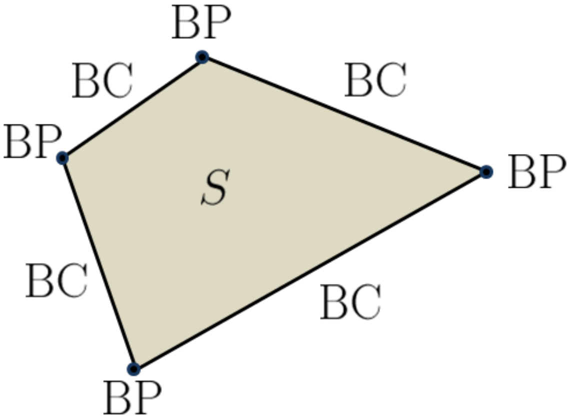
$$I = \int_0^1 \left[\int_{x^2}^{x^3} 3xy^2 \, dy \right] dx = \Big(\text{Treat } x \text{ as constant} \Big) = \int_0^1 \left[xy^3 \right]_{y=x^2}^{y=x^3} dx$$

$$\overset{FTC}{=} \int_0^1 \left[x\left(x^3\right)^3 - x\left(x^2\right)^3 \right] dx = \int_0^1 \left(x^{10} - x^7 \right) dx = \left[\frac{1}{11} x^{11} - \frac{1}{8} x^8 \right]_{x=0}^{x=1} = \boxed{-\frac{3}{88}}$$

Boundary Curves (BC's) & Boundary Points (BP's)

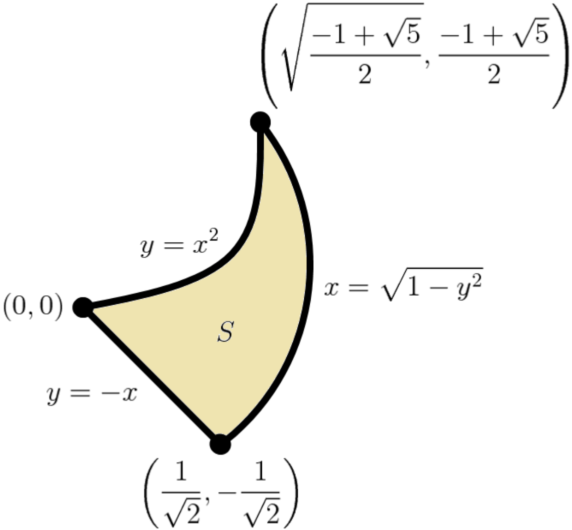


A **boundary point (BP)** is the **intersection** of two boundary curves (BC's).

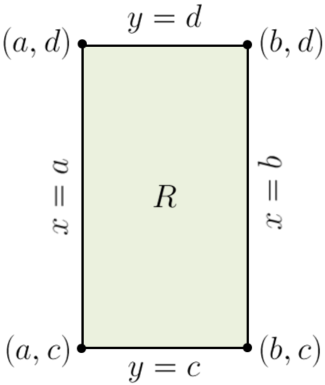


A **boundary point (BP)** is the **intersection** of two boundary curves (BC's).

Boundary Curves (BC's) & Boundary Points (BP's)



Area of a Rectangular Region using Iterated Integrals

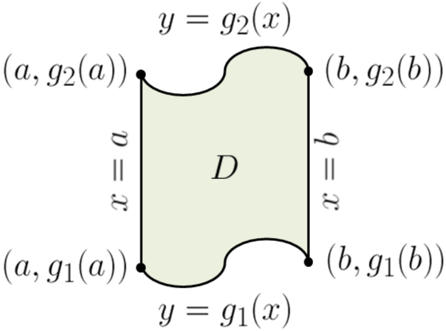


Rectangular Region

$$\text{Area}(R) := \iint_R dA = \int_c^d \int_a^b dx\, dy = \int_a^b \int_c^d dy\, dx$$

Rectangles are the simplest regions to double-integrate over.
(in fact, SST 12.1 exclusively covers double integrals over rectangles.)

Vertically Simple (V-Simple) Regions (Definition)



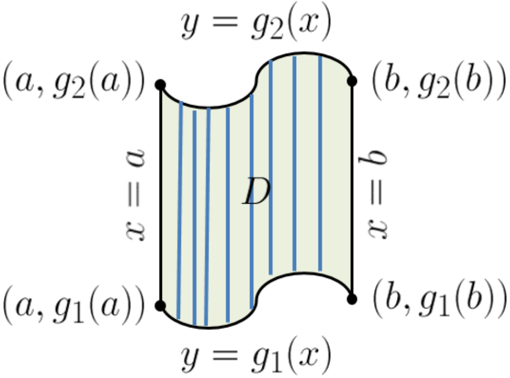
Vertically simple region

Definition

A region $D \subset \mathbb{R}^2$ is **vertically-simple (V-Simple)** if

the region has only one top BC & only one bottom BC.

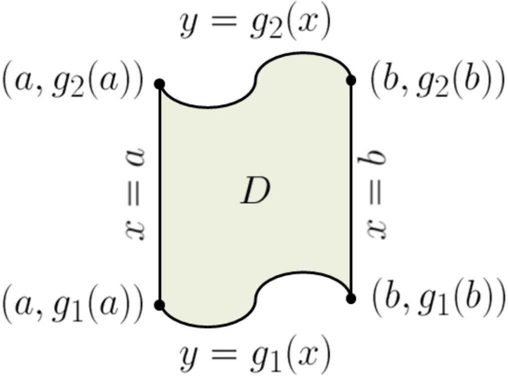
Vertically Simple (V-Simple) Regions (Definition)



Vertically simple region

i.e., V-Simple regions can be swept vertically (with vertical lines [in **blue**]) where each vertical line intersects the **same top BC** & **same bottom BC**.

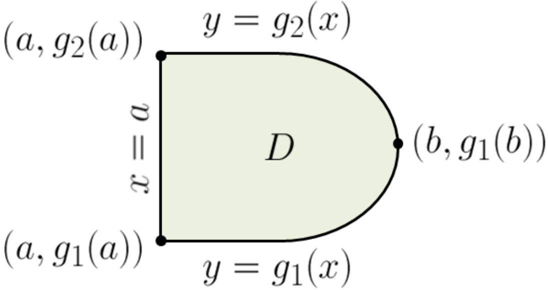
Area of a V-Simple Region using Iterated Integrals



Vertically simple region

$$\text{Area}(D) := \iint_D dA = \int_{\text{smallest } x\text{-value in } D}^{\text{largest } x\text{-value in } D} \int_{\text{bottom BC of } D}^{\text{top BC of } D} dy\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$$

Area of a V-Simple Region using Iterated Integrals



Vertically simple region

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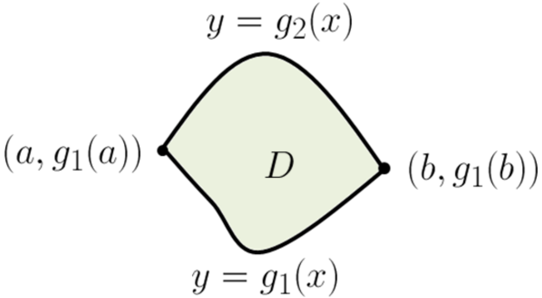

Vertically simple region

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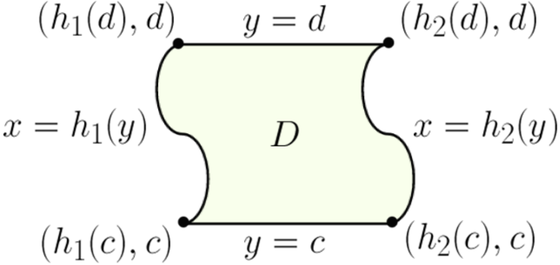
Horizontally Simple (H-Simple) Regions (Definition)



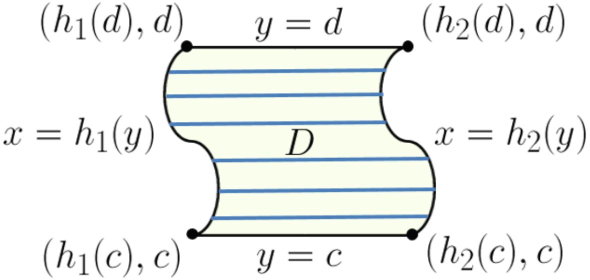
Horizontally simple region

Definition

A region $D \subset \mathbb{R}^2$ is **horizontally-simple (H-Simple)** if

the region has only one left BC & only one right BC.

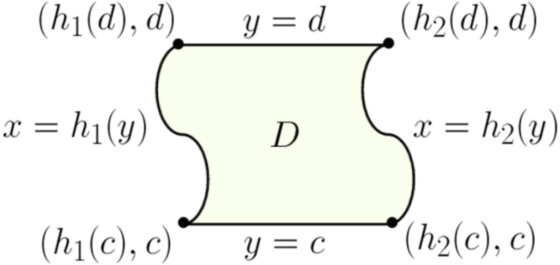
Horizontally Simple (H-Simple) Regions (Definition)



Horizontally simple region

i.e., H-Simple regions can be swept horizontally (w/ horizontal lines [in **blue**]) where each horizontal line intersects the **same left BC** & **same right BC**.

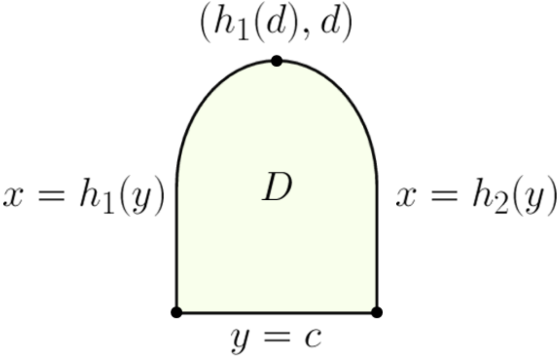
Area of a H-Simple Region using Iterated Integrals



Horizontally simple region

$$\text{Area}(D) := \iint_D dA = \int_{\text{smallest } y\text{-value in } D}^{\text{largest } y\text{-value in } D} \int_{\text{left BC of } D}^{\text{right BC of } D} dx\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy$$

Area of a H-Simple Region using Iterated Integrals



Horizontally simple region

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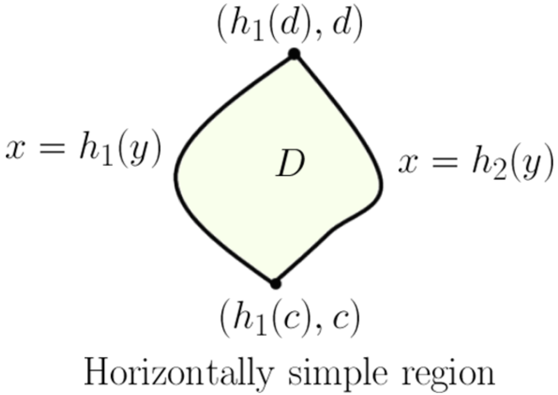
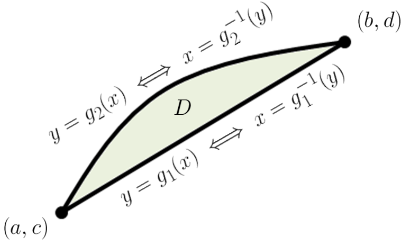
Area of a H-Simple Region using Iterated Integrals



$$\text{Area}(D) := \iint_D dA = \int_{\text{smallest } y\text{-value in } D}^{\text{largest } y\text{-value in } D} \int_{\text{left BC of } D}^{\text{right BC of } D} dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy$$

Area of a Region that's both V-Simple & H-Simple



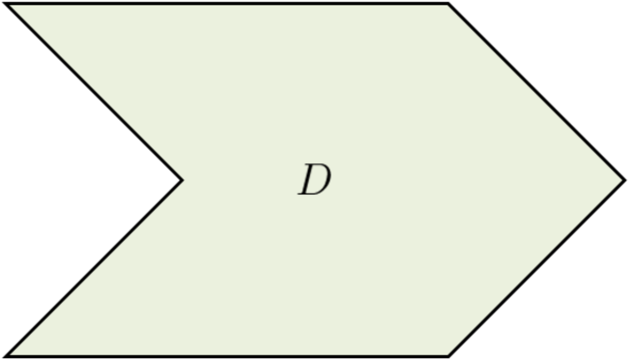
Both V-Simple & H-Simple Region

$$\text{Area}(D) := \iint_D dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx = \int_c^d \int_{g_2^{-1}(y)}^{g_1^{-1}(y)} dx\, dy$$

<u>REMARK:</u> The shape of region D or the integrand $f(x, y)$ may cause:

- Both integration orders $(dx\, dy)$ & $(dy\, dx)$ to yield Calc I integrals
- $(dx\, dy) \rightarrow$ nonelementary integral; $(dy\, dx) \rightarrow$ elementary integral
- $(dx\, dy) \rightarrow$ Calc II integral; $(dy\, dx) \rightarrow$ Calc I integral; or vice-versa

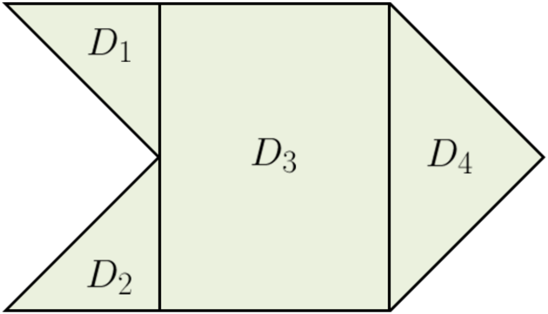
Area of a Region that's neither V-Simple nor H-Simple



Neither V-Simple Nor H-Simple Region

$$\text{Area}(D) := \iint_D dA = ?????$$

Area of a Region that's neither V-Simple nor H-Simple



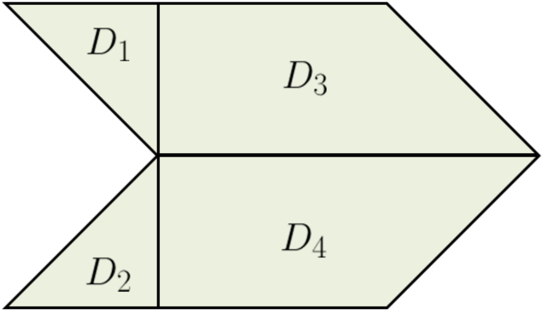
Subdivide Region into V-Simple and H-Simple Regions

$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

$$\text{Area}(D) := \iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA + \iint_{D_3} dA + \iint_{D_4} dA$$

Subdividing involves horizontal or vertical lines containing at least one BP.

Area of a Region that's neither V-Simple nor H-Simple



Subdivide Region into V-Simple and H-Simple Regions

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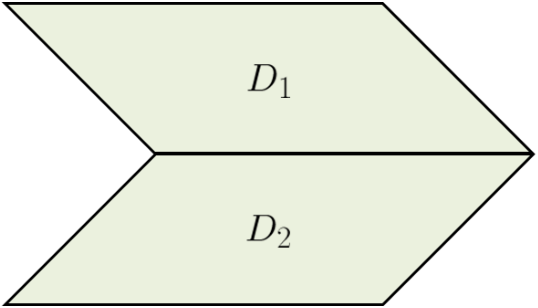
Subdividing involves horizontal or vertical lines containing at least one BP.

Area of a Region that's neither V-Simple nor H-Simple



Subdivide Region into V-Simple and H-Simple Regions

$$D = D_1 \cup D_2$$

$$\text{Area}(D) := \iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA$$

Subdividing involves horizontal or vertical lines containing at least one BP.

Proposition

Given closed & bounded region $D \subset \mathbb{R}^2$ on the xy-plane and solid $E \subset \mathbb{R}^3$ bounded below by the xy-plane & above by surface $z = f(x,y) \ \ \forall (x,y) \in D$:

0. Area$(D) = \iint_D dA$ Volume$(E) = \iint_D f(x,y)\, dA$

1. Sketch region D and label all BC's & BP's.

2. Determine whether D is V-Simple, H-Simple, Both, or Neither.

If D is neither V-Simple nor H-Simple, subdivide region.
Subdividing involves horizontal or vertical lines containing at least one BP.

3. Write appropriate iterated integral. (see previous slides)

BC $\equiv$ "**B**oundary **C**urve" BP $\equiv$ "**B**oundary **P**oint"

PART II:

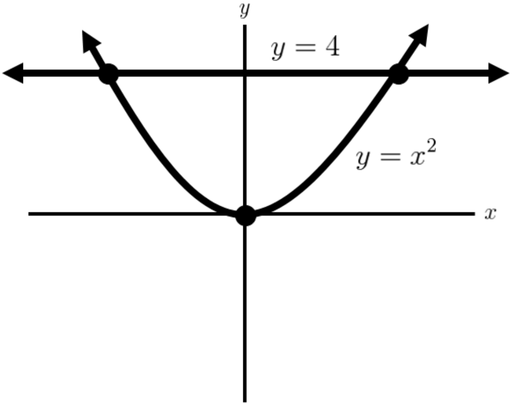
INTERCHANGING THE ORDER OF INTERGATION OF DOUBLE INTEGRALS

Interchanging the Order of Integration (Example)

WEX 12-2-3: Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.

Interchanging the Order of Integration (Example)

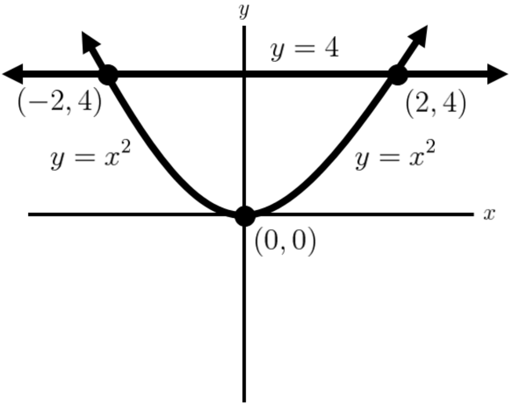
<u>**WEX 12-2-3:**</u> Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.



First, **sketch** the **region of integration**, label BC's.

Interchanging the Order of Integration (Example)

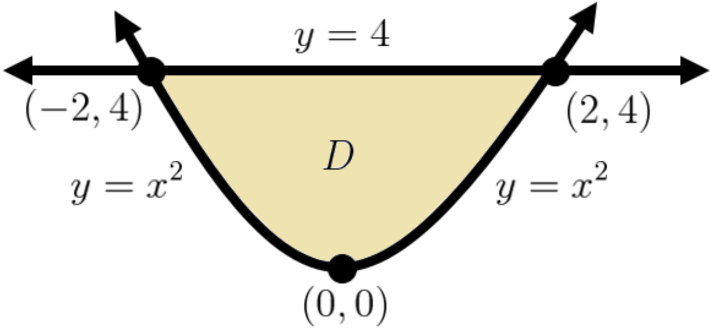
WEX 12-2-3: Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.



First, **sketch** the **region of integration**, label BC's & BP's.

Interchanging the Order of Integration (Example)

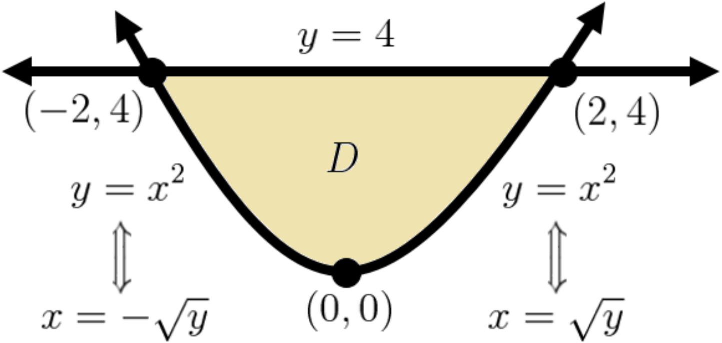
<u>WEX 12-2-3:</u> Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.



Remove clutter & label the **region of integration** D.

Interchanging the Order of Integration (Example)

WEX 12-2-3: Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.

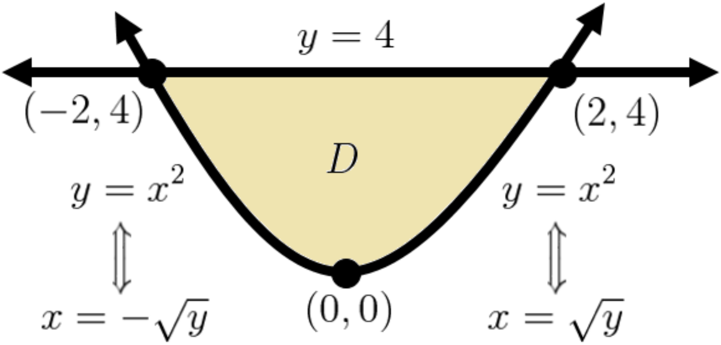


Label each BC as a function of y.

Notice that region D is **H-Simple**, so only one double integral is expected.

Interchanging the Order of Integration (Example)

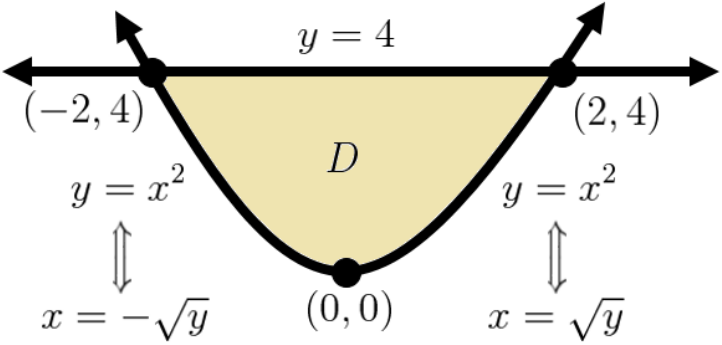
WEX 12-2-3: Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.



$$\therefore \quad I = \iint_D f\, dA = \int_{\text{smallest } y\text{-value in } D}^{\text{largest } y\text{-value in } D} \int_{\text{left BC of } D}^{\text{right BC of } D} f(x,y)\, dx\, dy$$

Interchanging the Order of Integration (Example)

WEX 12-2-3: Let $I = \int_{-2}^{2} \int_{x^2}^{4} f(x,y)\, dy\, dx$. Reverse the order of integration.



$$\therefore \quad I = \int_{\text{smallest } y\text{-value in } D}^{\text{largest } y\text{-value in } D} \int_{\text{left BC of } D}^{\text{right BC of } D} f(x,y)\, dx\, dy = \boxed{\int_{0}^{4} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx\, dy}$$

Fin.