Double Integrals: Polar Coordinates

Calculus III

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TTU

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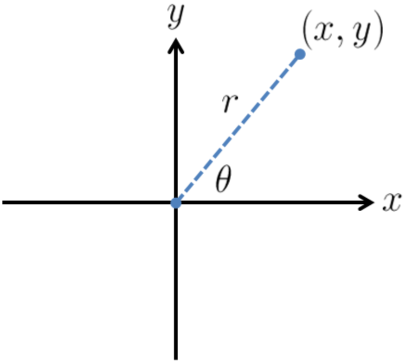
PART I:

SKETCHING POLAR CURVES

(REVIEW FROM CALCULUS II)

Convert: Rectangular Coord's $\leftrightarrow$ Polar Coord's



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

REMARK: r can be **negative**.

$\left(a \neq 0, b \neq 0, k \neq 0, n \in \mathbb{Z}_+\right)$ $\mathbb{Z}_+ \equiv$ The set of all **positive integers**.

POLAR CURVE	PROTOTYPE	REMARK(S)
Rays thru Pole	$\theta = k$	Always Graph!
Horizontal Lines (Off-Pole)	$r = a\csc\theta$	Always convert!
Vertical Lines (Off-Pole)	$r = a\sec\theta$	Always convert!
Circles Centered at Pole	$r = k$	Always Graph!
Circles Containing Pole	$r = a\cos\theta, r = a\sin\theta$	Always Graph!
Cardioids	$r = a \pm a\cos\theta, r = a \pm a\sin\theta$	Always Graph!
Limaçons	$r = b \pm a\cos\theta, r = b \pm a\sin\theta$	Always Graph!
Roses	$r = a\cos(n\theta), r = a\sin(n\theta)$	Always Graph!

The **pole** is the origin.

* See Section 6.3 of the textbook to see graphs of these types of curves.

$$r = a\csc\theta \iff r = \frac{a}{\sin\theta} \iff r\sin\theta = a \iff y = a$$

$$r = a\sec\theta \iff r = \frac{a}{\cos\theta} \iff r\cos\theta = a \iff x = a$$

Special Polar Curves NOT to be Considered

The following special polar curves are too subtle or complicated to graph and use with double integrals, and so will not be considered here: $\left(a \neq 0, b \neq 0, k \neq 0\right)$

- Lemniscates: $r^2 = a^2 \cos(2\theta), r^2 = a^2 \sin(2\theta)$
- Spirals: $r = k\theta, r = e^{\theta}, r = a^{k\theta}, r\theta = k$
- Strophoid: $r = a\cos(2\theta)\sec\theta$
- Bifolium: $r = a\sin\theta\cos^2\theta$
- Folium of Descartes: $r = \dfrac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$
- Ovals of Cassini: $r^4 + b^4 - 2b^2r^2\cos(2\theta) = k^4$

Graphing Polar Curves (Procedure)

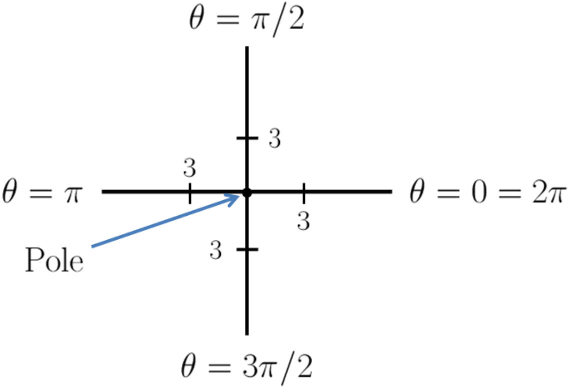
1. Graph $r = f(\theta)$ on the usual xy-plane where $x = \theta$ & $y = r$ (**Rectangular Plot**)
   Use special angles for θ: $\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$
   If $f(\theta)$ has a trig fcn, augument these angles accordingly & solve for θ:

   e.g. Suppose $f(\theta) = -5\cos\theta$.
   Then, since the argument of cosine is just θ,
   $\implies \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$

   e.g. Suppose $f(\theta) = 7\sin(2\theta)$.
   As always, let $\theta \in [0, 2\pi]$.
   Then, the argument of the sine fcn, $2\theta \in [0, 4\pi]$.
   $\implies 2\theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \frac{7\pi}{2}, 4\pi\right\}$
   $\implies \theta \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$
2. Use the rectangular plot of $r = f(\theta)$ to trace the polar graph of $r = f(\theta)$ (**Polar Plot**)

IMPORTANT: Except for equations of lines, ”connect the dots” using **smooth curves**, not line segments!

Polar Plots



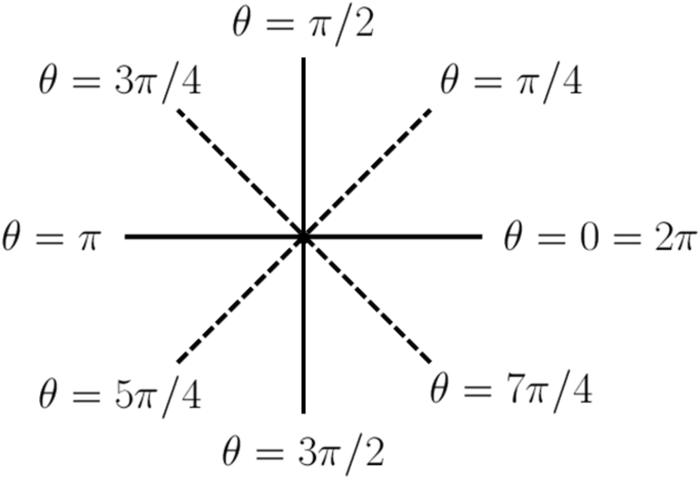
REMARKS:

Even though r can be negative, only label key **positive** r-values on each ray.

The **pole** is the origin, but it has no unique polar representation: $(0, \theta)$

Polar coordinates are NOT unique: $\left(2, \frac{7\pi}{4}\right) = \left(2, -\frac{\pi}{4}\right) = \left(-2, \frac{3\pi}{4}\right) = \left(-2, -\frac{5\pi}{4}\right)$

Polar Plots



If necessary, include more rays in the polar plot.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

Polar Plots (Example)

<u>**WEX 12-3-1:**</u> Plot the polar curve $r = -2\sin\theta$.

STEP 1: Identify the **key θ-values**.
The argument of the trig fcn should use ”easy angles”:

$$\implies \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$$

These are key values to label on the **horizontal axis** of **rectangular plot**.

Polar Plots (Example)

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

STEP 1: Identify the **key θ-values.**

The argument of the trig fcn should use "easy angles":

$$\implies \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$$

These are key values to label on the **horizontal axis** of **rectangular plot**.

STEP 2: Identify the **key r-values.**

Find the **range** of the curve: $\text{Rng}\Big[-2\sin\theta\Big] = [-2, 2]$

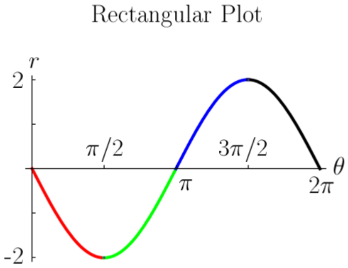
The key values are the curve's max value, min value, & mid value:

$$\text{Rng}\Big[-2\sin\theta\Big] = [-2, 2] \implies \begin{cases} \text{Max Value = 2} \\ \text{Min Value = -2} \end{cases}$$

$$\implies \text{Mid Value} := \frac{(\text{Max Value}) + (\text{Min Value})}{2} = \frac{2 + (-2)}{2} = 0$$

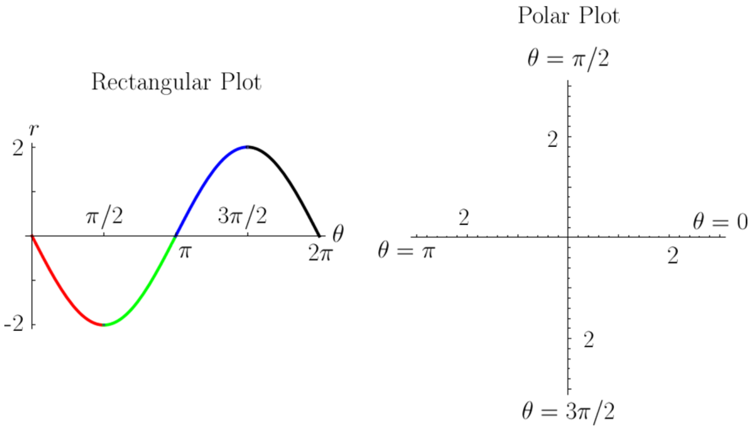
These are key values to label on the **vertical axis** of **rectangular plot**.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



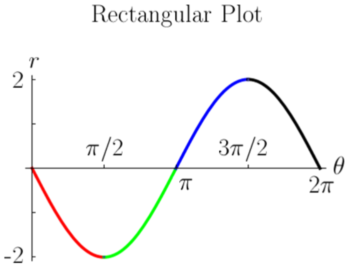
STEP 3: Trace the **rectangular plot** of $r = -2\sin\theta$

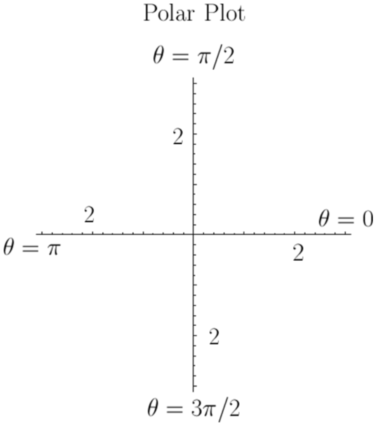
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



STEP 4: Trace the **polar plot**.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

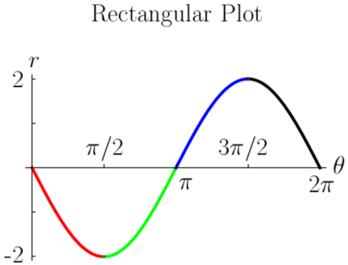


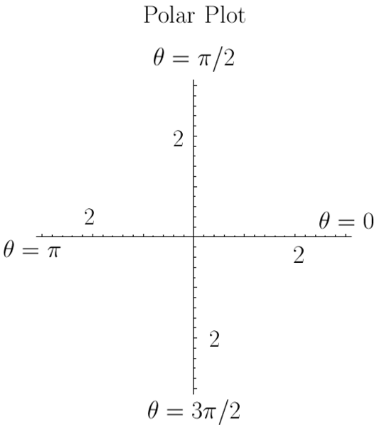


Setup the axes for the polar plot.

Polar Plots (Example)

<u>WEX 12-3-1:</u> Plot the polar curve $r = -2\sin\theta$.

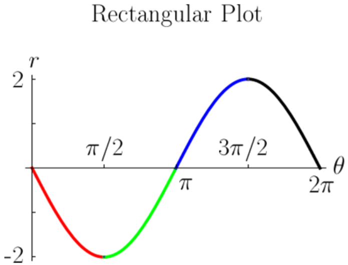


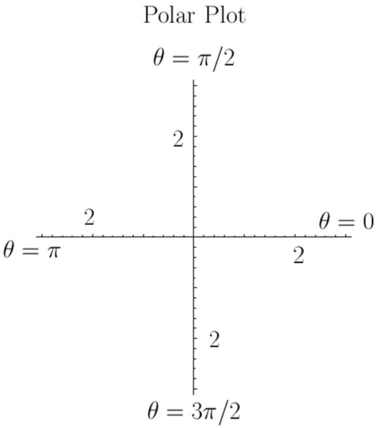


Consider the red portion of the rectangular plot (i.e. $\theta \in [0, \pi/2]$).

Polar Plots (Example)

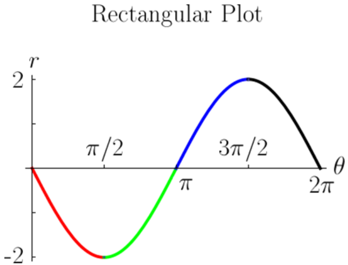
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

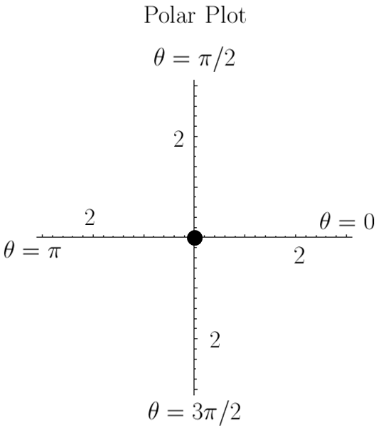




Starting Point: $\theta = 0 \implies r = 0$.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

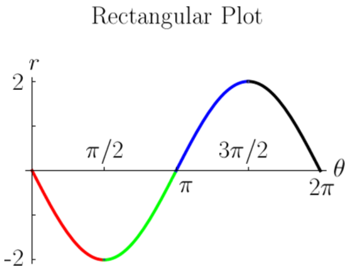


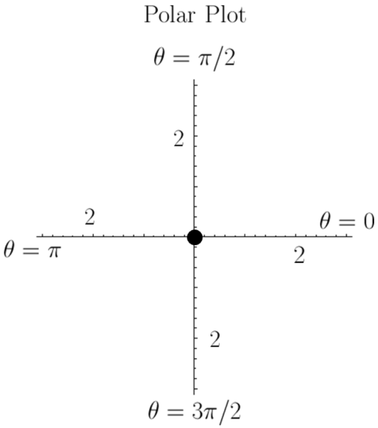


$$\theta = 0, r = 0 \implies \textbf{pole}.$$

Polar Plots (Example)

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

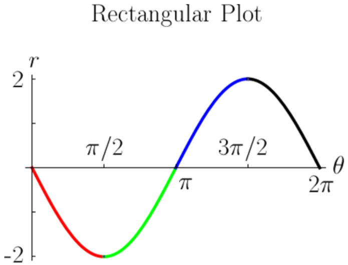


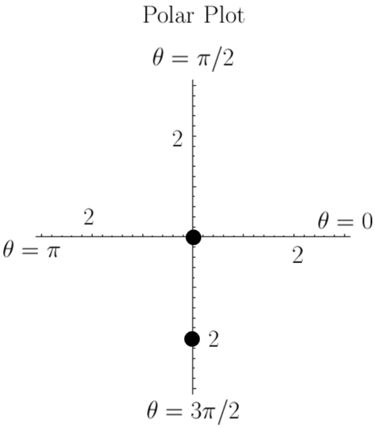


Ending Point: $\theta = \pi/2 \implies r = -2$.

Polar Plots (Example)

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

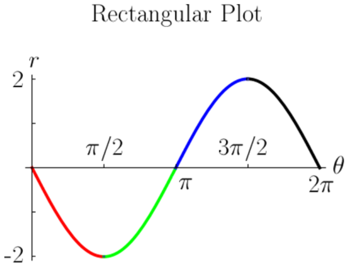


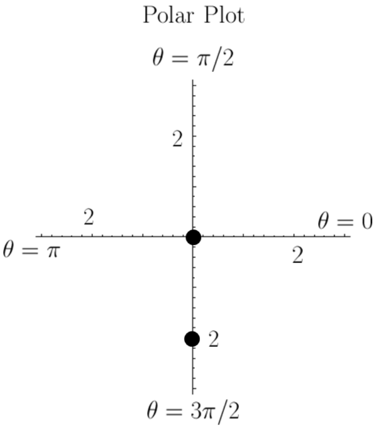


$\theta = \pi/2, r = -2 \implies$ March 2 units in **opposite direction** from $\theta = \pi/2$.

Polar Plots (Example)

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

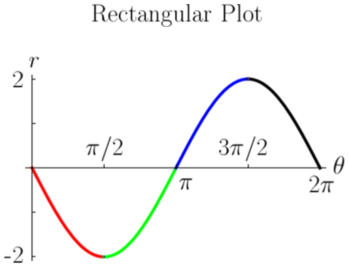


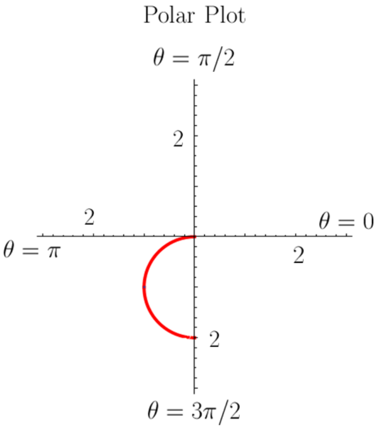


In Between: For $\theta \in (0, \pi/2) \subseteq$ QI, $r < 0$ and r **departs from zero**.

Polar Plots (Example)

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

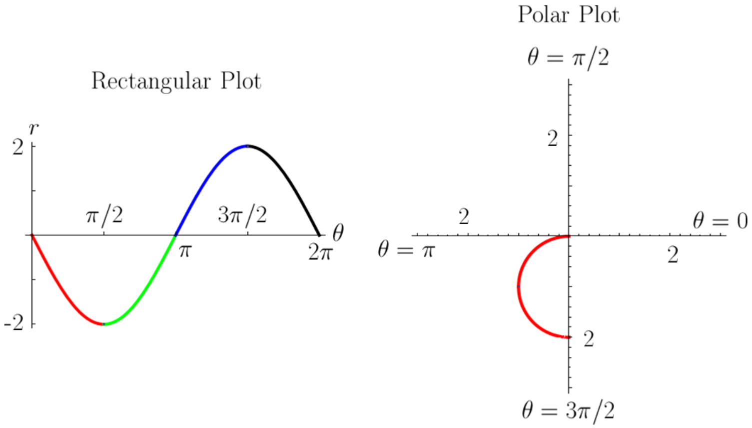




In Between: Trace a smooth curve in QIII that **departs from pole**.

Polar Plots (Example)

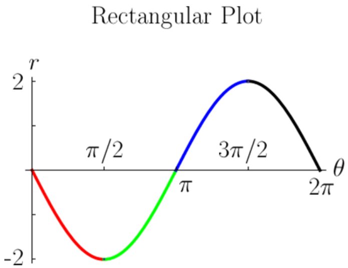
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

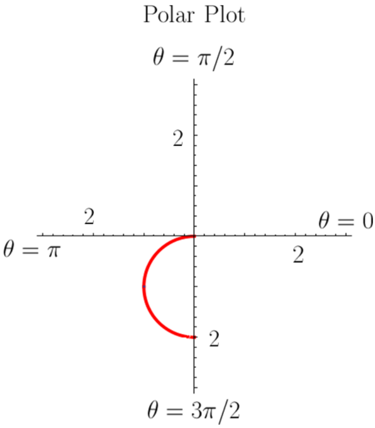


Consider the green portion of the rectangular plot (i.e. $\theta \in [\pi/2, \pi]$).

Polar Plots (Example)

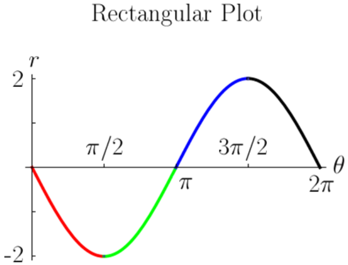
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

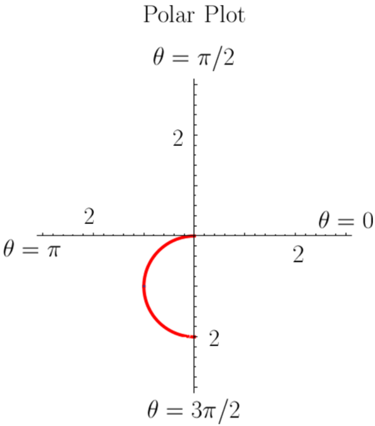




Starting Point: $\theta = \pi/2 \implies r = -2$.

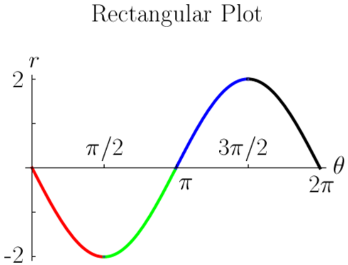
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

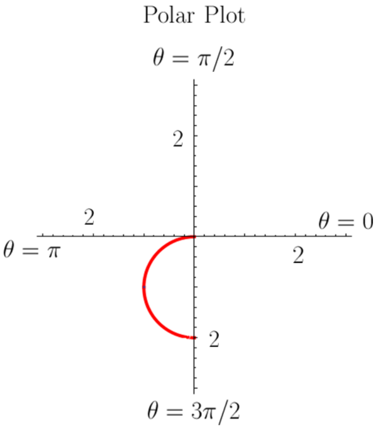




$\theta = \pi/2, r = -2 \implies$ March 2 units in **opposite direction** from $\theta = \pi/2$.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

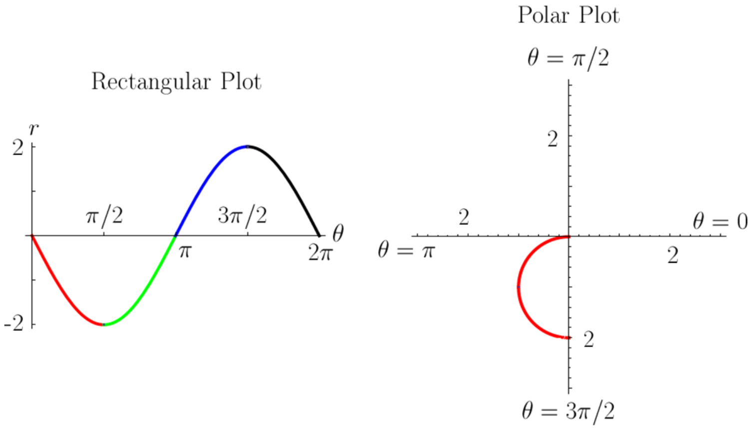




Ending Point: $\theta = \pi \implies r = 0$.

Polar Plots (Example)

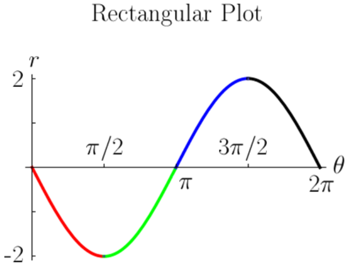
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

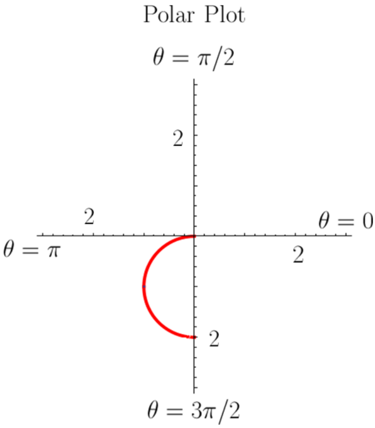


$$\theta = \pi, r = 0 \implies \textbf{pole}.$$

Polar Plots (Example)

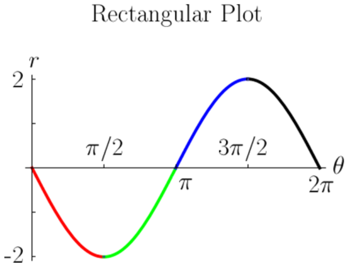
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

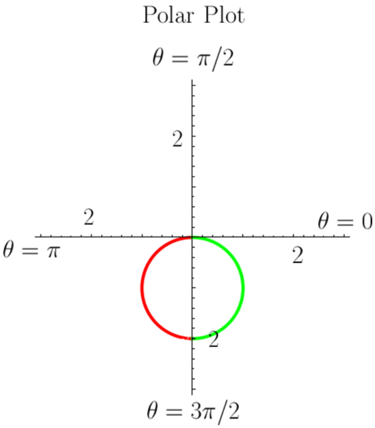




In Between: For $\theta \in (\pi/2, \pi) =$ QII, $r < 0$ and r **approaches zero**.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

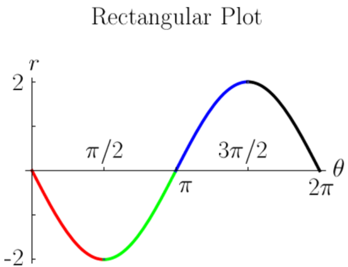


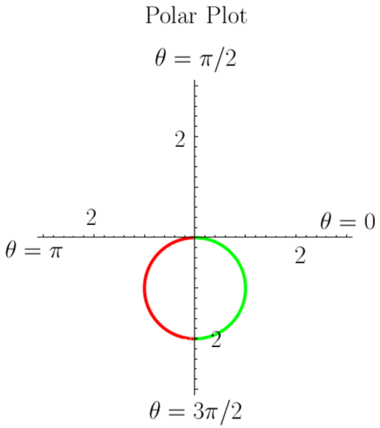


In Between: Trace a smooth curve in QIV that **approaches pole**.

Polar Plots (Example)

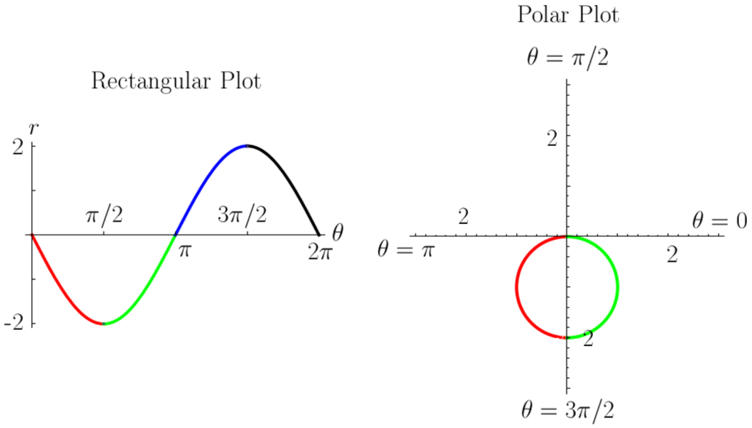
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.





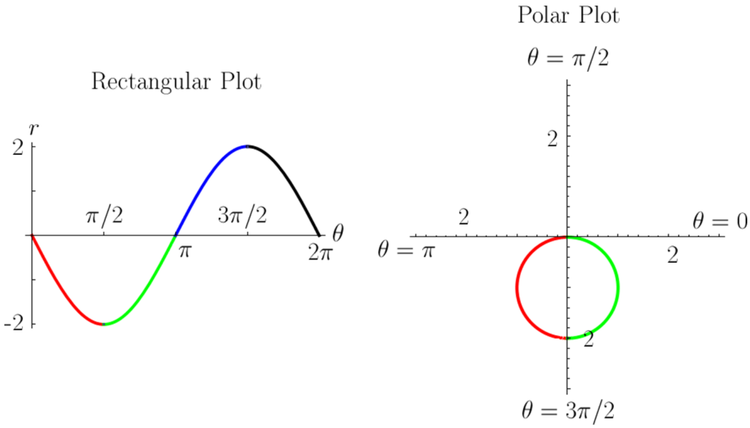
AND WE'RE DONE!

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



Done? Already? But why?? What about $\theta \in [\pi, 2\pi]$??

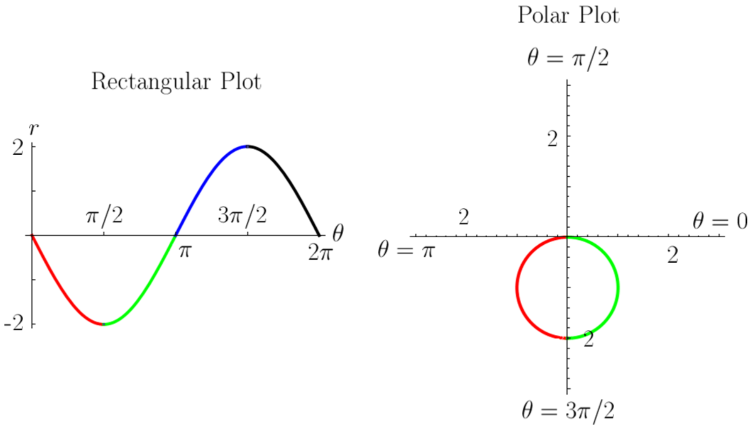
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



Because continuing further would **re-trace the same curve**!

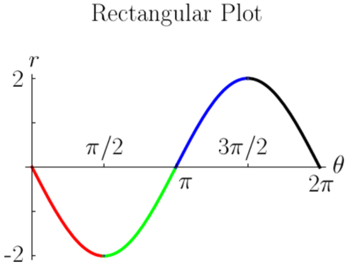
Polar Plots (Example)

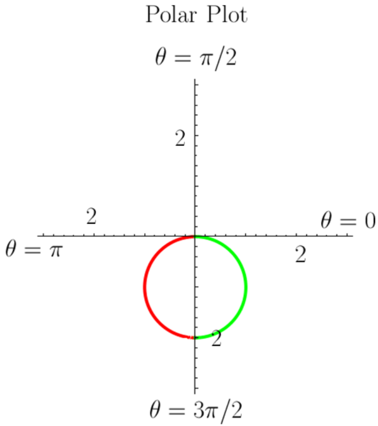
<u>WEX 12-3-1:</u> Plot the polar curve $r = -2\sin\theta$.



But let's continue anyway to see why the same curve is re-traced.....

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

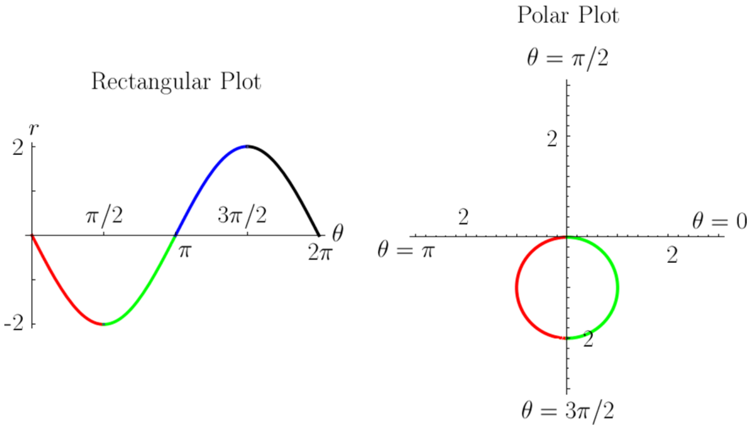




Consider the blue portion of the rectangular plot (i.e. $\theta \in [\pi, 3\pi/2]$).

Polar Plots (Example)

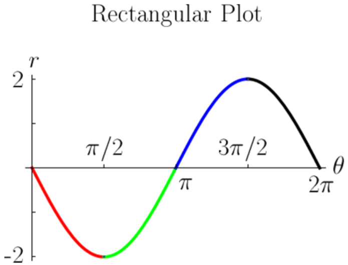
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

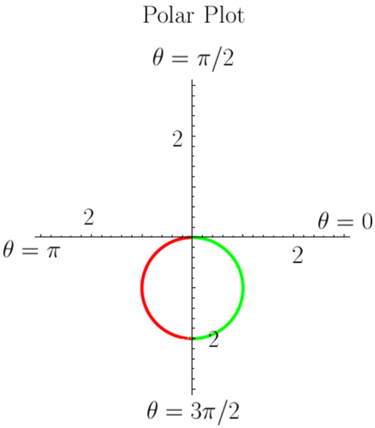


Starting Point: $\theta = \pi \implies r = 0$.

Polar Plots (Example)

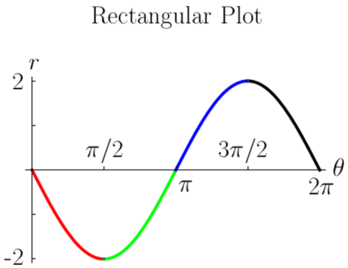
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

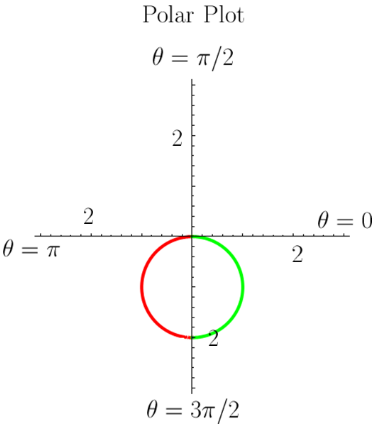




$$\theta = \pi, r = 0 \implies \textbf{pole}.$$

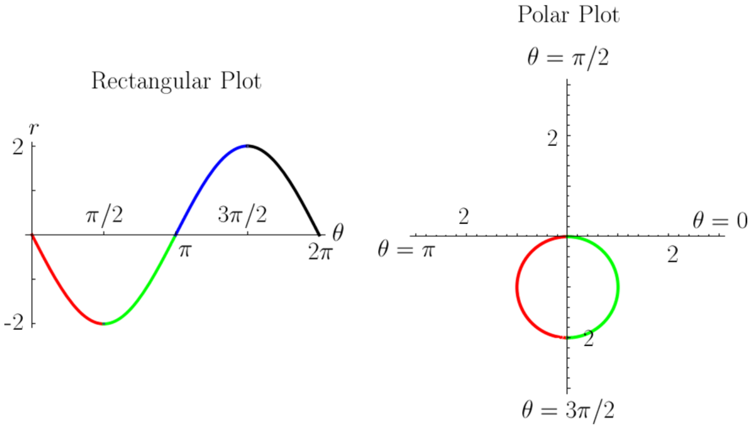
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.





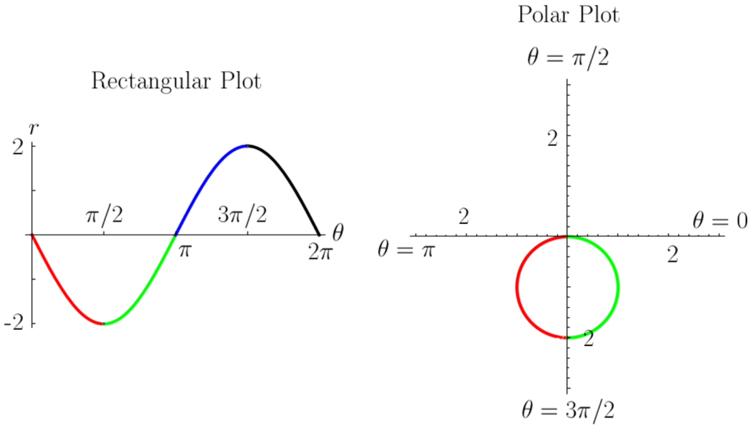
Ending Point: $\theta = 3\pi/2 \implies r = 2$.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



$\theta = 3\pi/2, r = 2 \implies$ March 2 units in **same direction** as $\theta = 3\pi/2$.

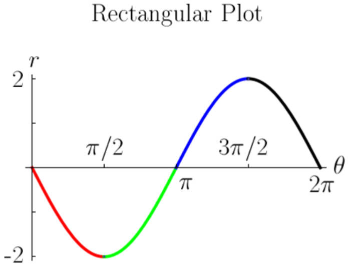
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

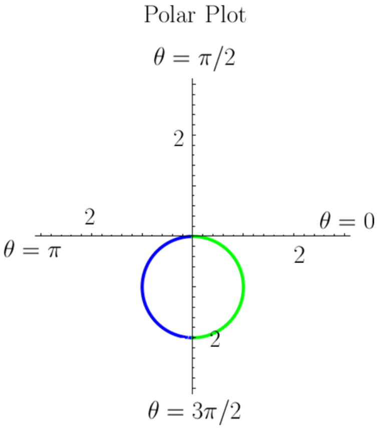


In Between: For $\theta \in (\pi, 3\pi/2) \subseteq$ QIII, $r > 0$ and r **departs from zero**.

Polar Plots (Example)

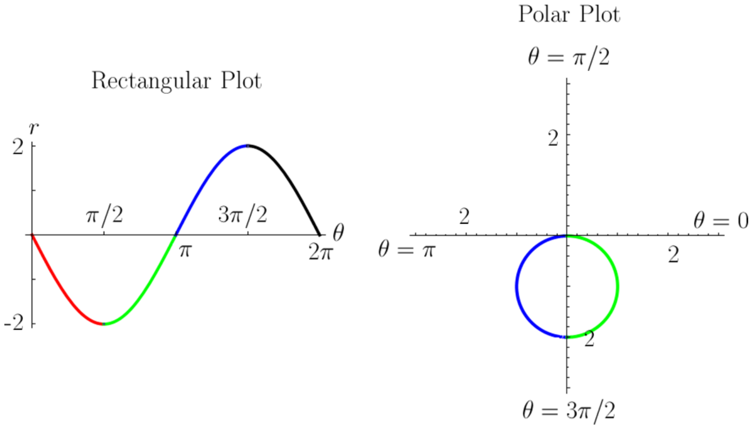
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.





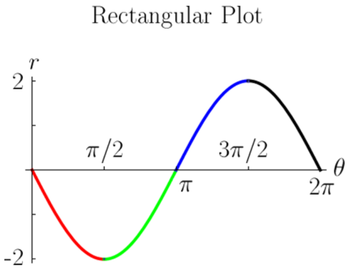
In Between: Trace a smooth curve in QIII that **departs from pole**.

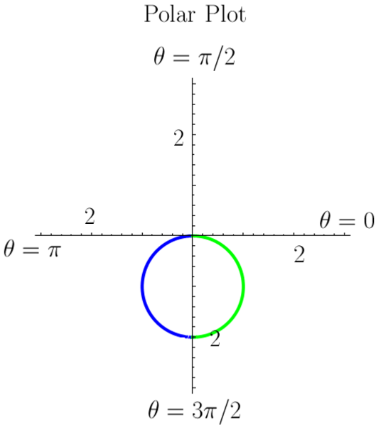
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



Consider the black portion of the rectangular plot (i.e. $\theta \in [3\pi/2, 2\pi]$).

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

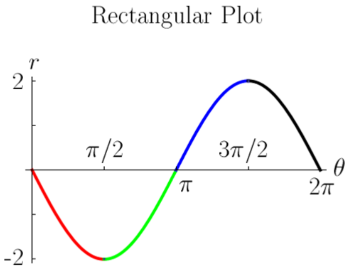


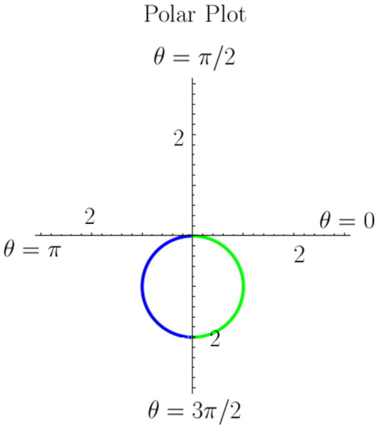


Starting Point: $\theta = 3\pi/2 \implies r = 2$.

Polar Plots (Example)

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

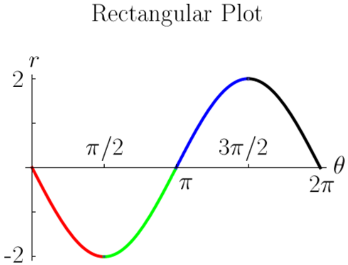


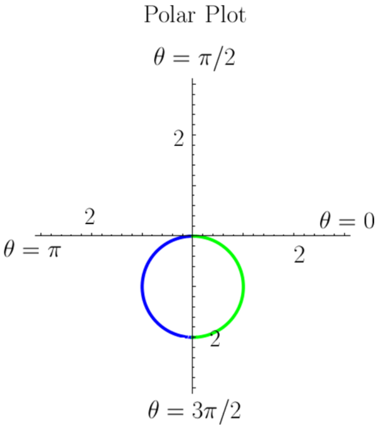


$\theta = 3\pi/2, r = 2 \implies$ March 2 units in **same direction** as $\theta = 3\pi/2$.

Polar Plots (Example)

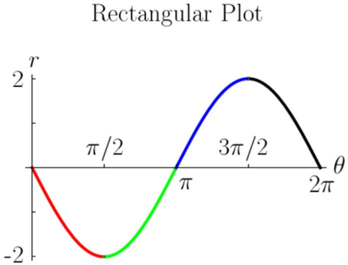
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

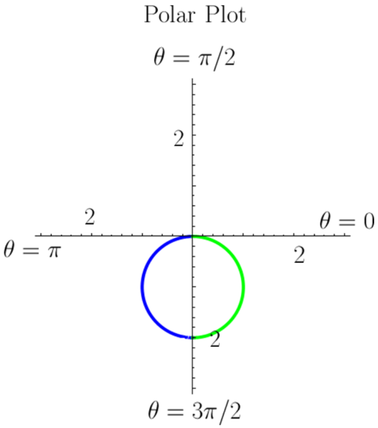




Ending Point: $\theta = 2\pi \implies r = 0$.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

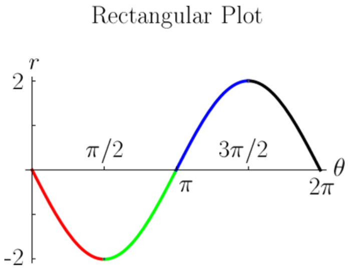


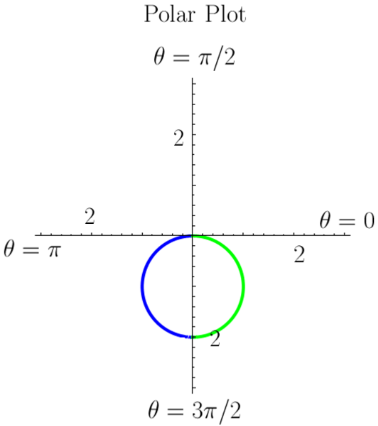


$$\theta = 2\pi, r = 0 \implies \textbf{pole}.$$

Polar Plots (Example)

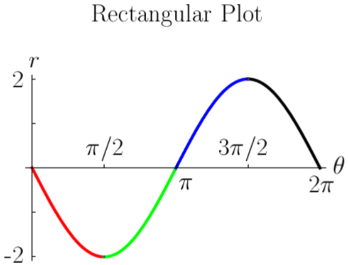
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.

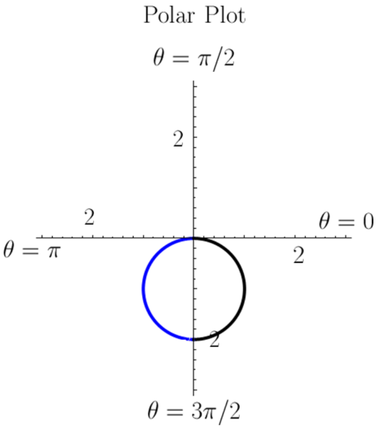




In Between: For $\theta \in (3\pi/2, 2\pi) \subseteq$ QIV, $r > 0$ and r **approaches zero**.

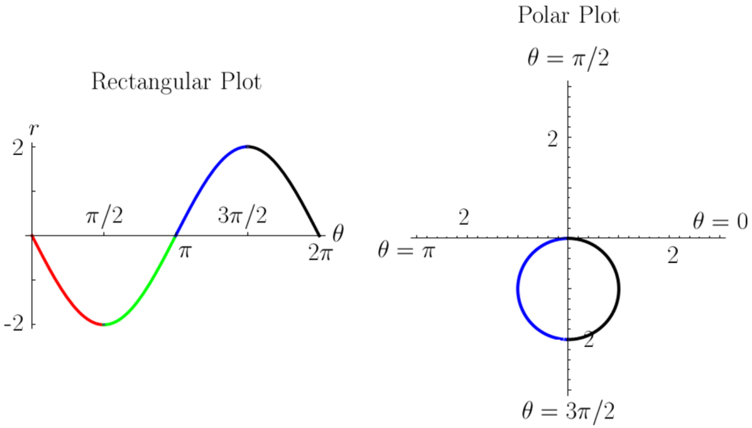
WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.





In Between: Trace a smooth curve in QIV that **approaches pole**.

WEX 12-3-1: Plot the polar curve $r = -2\sin\theta$.



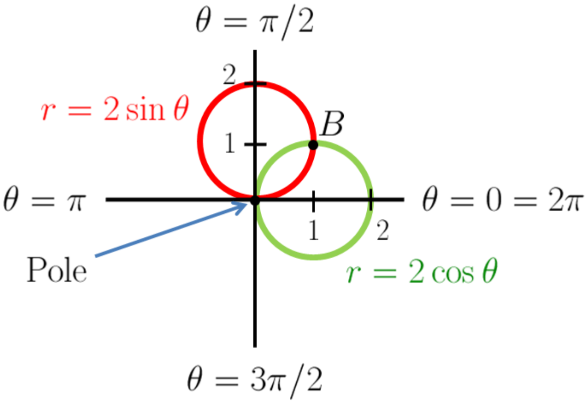
Moral: NEVER RE-TRACE THE SAME POLAR CURVE!

Intersection of Two Polar Curves

TASK: Find all intersection points of the polar curves $r = f(\theta)$ & $r = g(\theta)$.

- Solving $f(\theta) = g(\theta)$ finds some, but not necessarily all, intersection points.
- In particular, intersections at the **pole** are nearly impossible to find algebraically because the pole has no single representation in polar coordinates that satisfies both $r = f(\theta)$ & $r = g(\theta)$.
- Therefore, to find all intersection points, **graph both curves**!

Intersection of Two Polar Curves (Example)

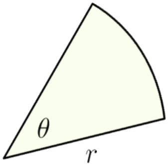


$2\sin\theta = 2\cos\theta \implies \cos\theta(\tan\theta - 1) = 0 \implies \cos\theta = 0$ or $\tan\theta = 1$
$\implies \theta \in \{\pi/2, 3\pi/2\}$ or $\theta \in \{\pi/4, 5\pi/4\}$
Discard $\theta = \pi/2$ & $\theta = 3\pi/2$ since they're **extraneous solutions**.
Both $\theta = \pi/4$ and $\theta = 5\pi/4$ yield the same intersection point B.
So solving algebraically did not yield the intersection point at the pole.

PART II:

AREAS OF POLAR REGIONS
(i.e. AREAS OF REGIONS BOUNDED BY POLAR CURVES)

Area of a Sector



Proposition

The area of a sector with radius r that sweeps an angle θ is:

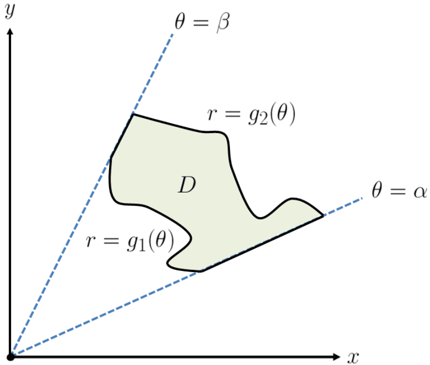
$$\textit{Area of Sector}(r, \theta) = \frac{1}{2} r^2 \theta$$

<u>PROOF:</u> Area of Circle$(r) = \pi r^2$.
But a circle is just a sector that sweeps an angle of 2π radians.

$$\implies \text{Area of Sector}(r, \theta) = \left(\frac{\theta}{2\pi}\right) \Big[\text{Area of Circle}(r)\Big] = \left(\frac{\theta}{2\pi}\right) (\pi r^2) = \frac{1}{2} r^2 \theta$$

QED

Radially-Simple (r-Simple) Polar Regions (Definition)



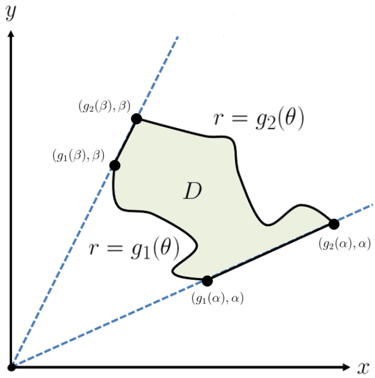
Above, the **inner BC** is $r = g_1(\theta)$ & the **outer BC** is $r = g_2(\theta)$.
Notice, the **inner BC** & **outer BC** are both traced for $\theta \in [\alpha, \beta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **radially-simple (r-Simple)** if

D has one **inner BC** & one **outer BC**, both traced over the <u>same</u> θ-values.

Radially-Simple (r-Simple) Polar Regions (Definition)



Above, the **inner BC** is $r = g_1(\theta)$ & the **outer BC** is $r = g_2(\theta)$.
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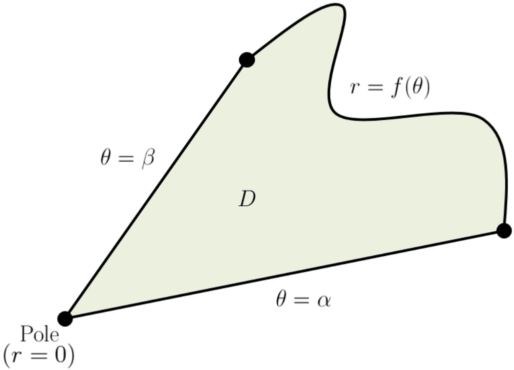
Radially-Simple (r-Simple) Polar Regions (Definition)



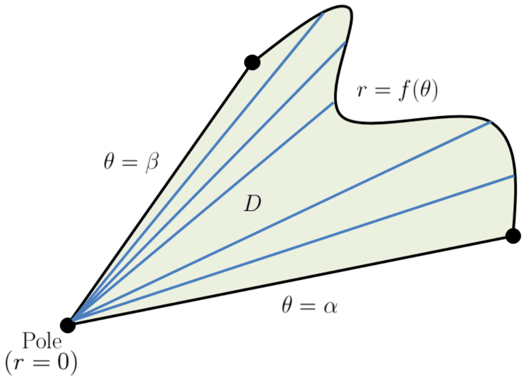
Above, the **inner BC** is the **pole** ($r = 0$) & the **outer BC** is $r = f(\theta)$.
Notice, the **outer BC** is traced for $\theta \in [\alpha, \beta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **radially-simple (r-Simple)** if

D has the **pole** as the **inner BC** & only one **outer BC**.

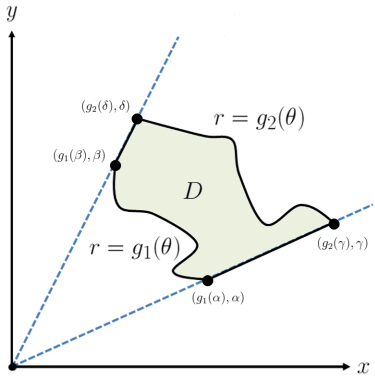
Radially-Simple (r-Simple) Polar Regions (Definition)



Above, the **inner BC** is the **pole** $(r = 0)$ & the **outer BC** is $r = f(\theta)$.

i.e., r-Simple regions can be swept radially from the pole (with rays [in **blue**]) where each ray enters the **same inner BC** & exits the **same outer BC**.

Quasi-Radially-Simple (Quasi-r-Simple) Polar Region



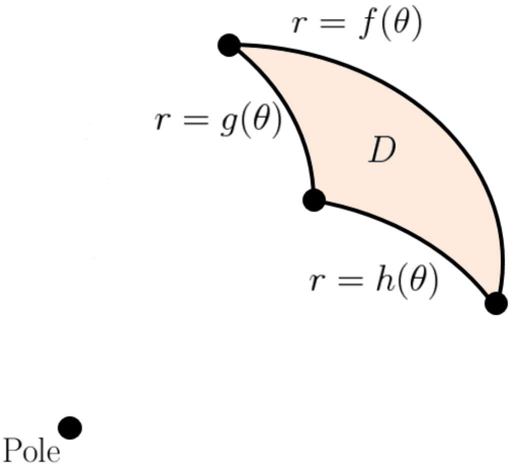
Above, the **inner BC** is $r = g_1(\theta)$ & the **outer BC** is $r = g_2(\theta)$.
Notice, **inner BC** is traced for $\theta \in [\alpha, \beta]$ & **outer BC** is traced for $\theta \in [\gamma, \delta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **quasi-radially-simple (Quasi-r-Simple)** if

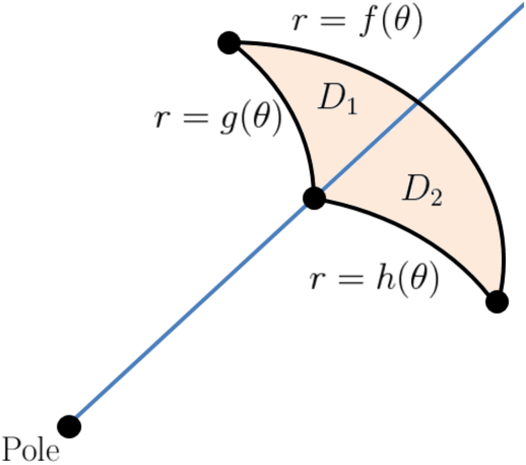
D has one **inner BC** & one **outer BC**, each traced over <u>different</u> θ-values.

Region that's neither r-Simple nor Quasi-r-Simple



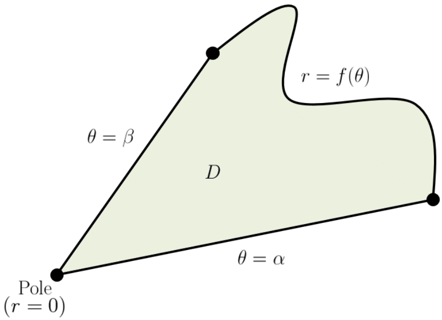
How to handle a polar region that lacks (quasi-) radial simplicity???

Region that's neither r-Simple nor Quasi-r-Simple



Subdivide polar region **radially from the pole** through appropriate BP.

Area of a Radially Simple (r-Simple) Polar Region

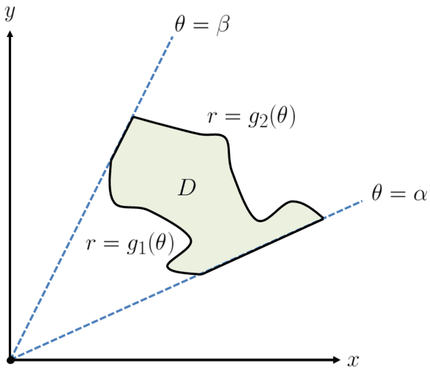


Proposition

Let D be a r-simple region as shown above. Then:

$$\textit{Area}(D) = \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Pole}}^{\text{Outer BC of } D} r\, dr\, d\theta = \int_{\alpha}^{\beta} \int_{0}^{f(\theta)} r\, dr\, d\theta$$

Area of a Radially Simple (r-Simple) Polar Region

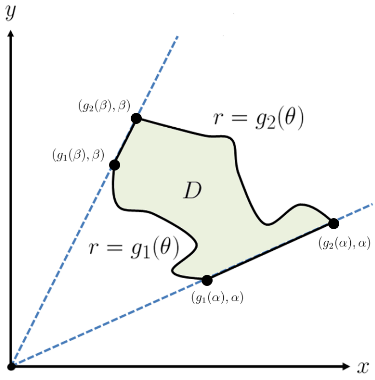


Proposition

Let D be a r-simple region as shown above. Then:

$$\text{Area}(D) = \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Inner BC of } D}^{\text{Outer BC of } D} r\,dr\,d\theta = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta$$

Area of a Radially Simple (r-Simple) Polar Region

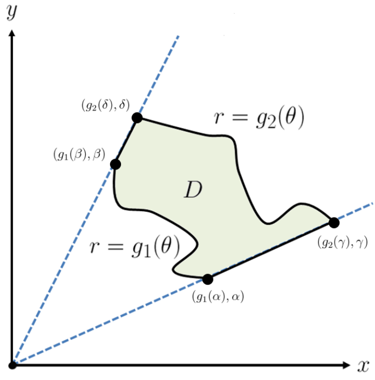


Proposition

Let D be a r-simple region as shown above. Then:

$$\textit{Area}(D) = \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Inner BC of } D}^{\text{Outer BC of } D} r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} r \, dr \, d\theta$$

Area of a Quasi-r-Simple Polar Region

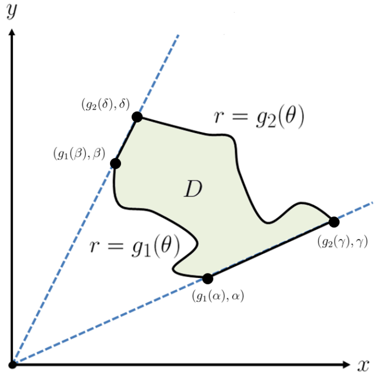


Proposition

Let D be a quasi-r-simple region as shown above. Then Area$(D) =$

$$\int_{\text{Smallest } \theta \text{ for Outer BC}}^{\text{Largest } \theta \text{ for Outer BC}} \int_{\text{Pole}}^{\text{Outer BC}} r\,dr\,d\theta - \int_{\text{Smallest } \theta \text{ for Inner BC}}^{\text{Largest } \theta \text{ for Inner BC}} \int_{\text{Pole}}^{\text{Inner BC}} r\,dr\,d\theta$$

Area of a Quasi-r-Simple Polar Region



Proposition

Let D be a quasi-r-simple region as shown above. Then:

$$\text{Area}(D) = \int_{\gamma}^{\delta} \int_{0}^{g_2(\theta)} r \, dr \, d\theta - \int_{\alpha}^{\beta} \int_{0}^{g_1(\theta)} r \, dr \, d\theta$$

Converting a Double Integral from Rectangular to Polar Coordinates

Recall how to convert rectangular coord's $\to$ polar coord's: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Proposition

Let $f(x,y) \in C(D)$
where D *is either a* r*-simple or quasi-*r*-simple region (or can subdivided).*
Then

$$\iint_D f(x,y)\, dA = \iint_D f(r\cos\theta, r\sin\theta)\; r\; dr\; d\theta$$

NEVER SETUP DOUBLE INTEGRALS IN THE ORDER $r\,d\theta\,dr$!

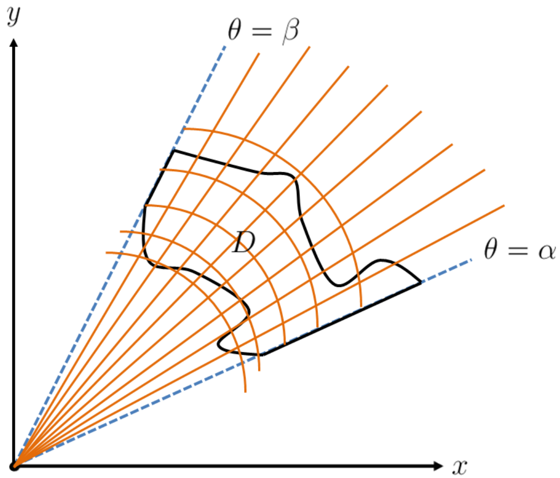
For the purposes of this course:
NEVER, EVER, EVER, SETUP A DOUBLE INTEGRAL IN POLAR FORM IN THE ORDER $r\,d\theta\,dr$, SUCH AS $\int\int f(r,\theta)\,r\,d\theta\,dr$, ...

.....EVER!!!!!!!

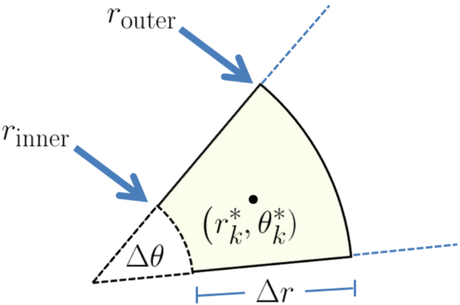
Uh...Okay....Why Not??? Two reasons:

- All non-ray polar curves are expressed as $r = f(\theta)$, not as $\theta = g(r)$.
- Sweeping a region by angle first causes way too many subtleties to occur.

Why $\int\int r\,dr\,d\theta$ instead of just $\int\int dr\,d\theta$?? (Optional)



Subdivide region D into N **polar rectangles**.

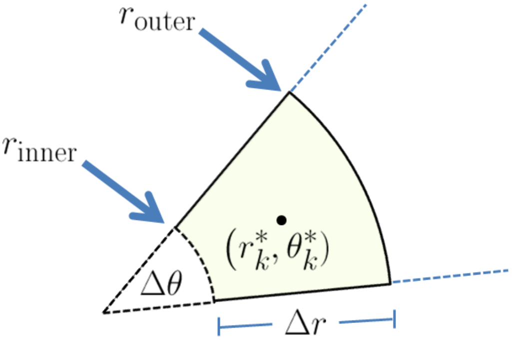


k^{th} Polar Rectangle

$$\Delta A_k = \Big(\text{Area of Large Sector}\Big) - \Big(\text{Area of Small Sector}\Big)$$

Let (r_k^*, θ_k^*) be the **center point** of the k^{th} polar rectangle.

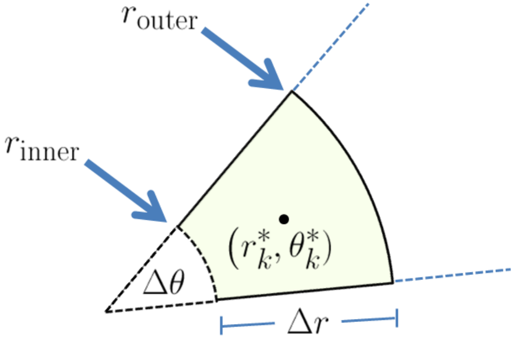
Why $\int\int r\,dr\,d\theta$ instead of just $\int\int dr\,d\theta$?? (Optional)



k^{th} Polar Rectangle

$$\Delta A_k = \frac{1}{2} r_{\text{outer}}^2 \Delta\theta - \frac{1}{2} r_{\text{inner}}^2 \Delta\theta$$

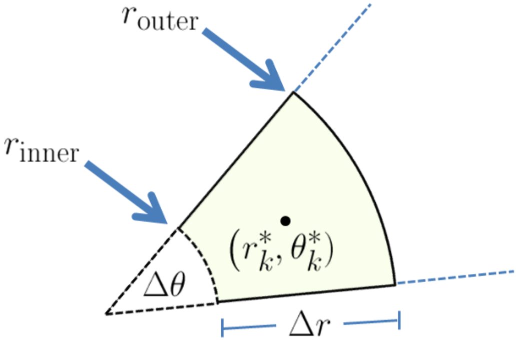
Why $\int\int r\,dr\,d\theta$ instead of just $\int\int dr\,d\theta$?? (Optional)



k^{th} Polar Rectangle

$$\Delta A_k = \frac{1}{2}(r_{\text{outer}} + r_{\text{inner}})(r_{\text{outer}} - r_{\text{inner}})\Delta\theta$$

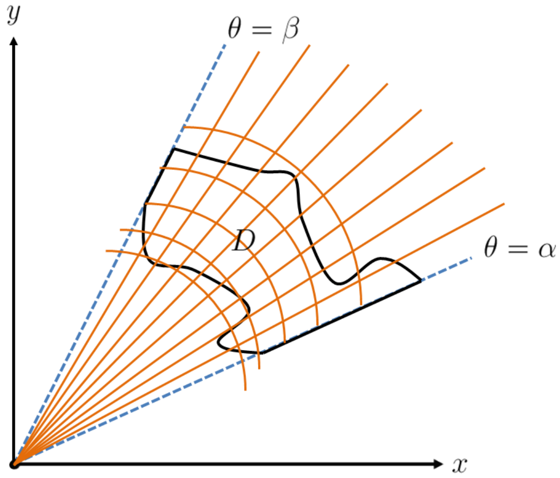
Why $\int\int r\,dr\,d\theta$ instead of just $\int\int dr\,d\theta$?? (Optional)



k^{th} Polar Rectangle

$$\Delta A_k = r_k^* \Delta r \Delta\theta$$

Why $\int\int r\,dr\,d\theta$ instead of just $\int\int dr\,d\theta$?? (Optional)



$$\text{Area}(D) := \iint_D dA = \lim_{N\to\infty} \sum_{k=1}^{N} \Delta A_k = \lim_{N\to\infty} \sum_{k=1}^{N} r_k^* \Delta r \Delta\theta = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta$$

Fin.