Triple Integrals

Calculus III

Josh Engwer

TTU

22 October 2014

Triple Integrals (Definition & Geometric Interpretation)

Definition

(Riemann Sum Definition of the Triple Integral)

Let $E \subset \mathbb{R}^3$ be a closed and bounded **solid** and $f(x, y, z) \in C(E)$.
Then the **triple integral of** f over E is defined to be

$$\iiint_E f \, dV \equiv \iiint_E f(x, y, z) \, dV := \lim_{N \to \infty} \sum_{k=1}^{N} f\left(x_k^*, y_k^*, z_k^*\right) \Delta V$$

Proposition

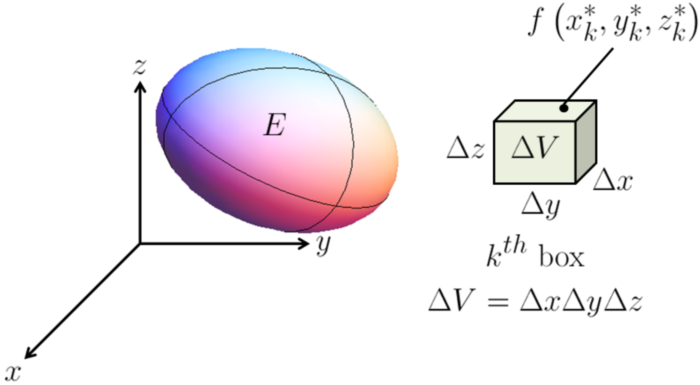
(The Triple Integral as a Volume)

Let $E \subset \mathbb{R}^3$ be a closed and bounded ***solid****.*
Then the ***volume of the solid*** *E is defined to be*

$$\iiint_E dV$$

Triple Integrals (Riemman Sum Definition)

Partition solid E into N boxes, each with volume ΔV



$$\iiint_E f\, dV := \lim_{N\to\infty} \sum_{k=1}^{N} f\left(x_k^*, y_k^*, z_k^*\right) \Delta V$$

Triple Integrals (Properties)

Let set E be a closed & bounded solid in $\mathbb{R}^3$.
Let functions $f(x,y,z)$ & $g(x,y,z)$ be defined & continuous on E.
Let $k \in \mathbb{R}$.

Constant Multiple Rule: $\iiint_E kf\ dV = k \iiint_E f\ dV$

Sum/Difference Rule: $\iiint_E \left[f \pm g\right]\ dV = \iiint_E f\ dV \pm \iiint_E g\ dV$

Nonnegativity Rule: $f(x,y,z) \geq 0\ \ \forall (x,y,z) \in E \implies \iiint_E f\ dV \geq 0$

Dominance Rule:
$f(x,y,z) \leq g(x,y,z)\ \ \forall (x,y,z) \in E \implies \iiint_E f\ dV \leq \iiint_E g\ dV$

Solid Additivity Rule:
$E = E_1 \cup E_2 \implies \iiint_E f\ dV = \iiint_{E_1} f\ dV + \iiint_{E_2} f\ dV$

Triple Integrals written as Iterated Integrals

Using the Riemann Sum Def'n to compute $\iiint_E f\, dV$ is too tedious & hard!

Instead write the triple integral as an **iterated integral**:

$$\iiint_E f\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

OR

$$\iiint_E f\, dV = \int_c^d \int_{g_3(y)}^{g_4(y)} \int_{h_3(y,z)}^{h_4(y,z)} f(x,y,z)\, dx\, dz\, dy$$

OR

$$\iiint_E f\, dV = \int_p^q \int_{g_5(z)}^{g_6(z)} \int_{h_5(x,z)}^{h_6(x,z)} f(x,y,z)\, dy\, dx\, dz$$

To compute an iterated integral, compute the "inner integral" first, then compute the "middle integral", then compute the "outer integral."

Triple Integrals written as Iterated Integrals

Using the Riemann Sum Def'n to compute $\iiint_E f\, dV$ is too tedious & hard!

Instead write the triple integral as an **iterated integral**:

$$\iiint_E f\, dV = \int_c^d \int_{g_7(y)}^{g_8(y)} \int_{h_7(x,y)}^{h_8(x,y)} f(x,y,z)\, dz\, dx\, dy$$

OR

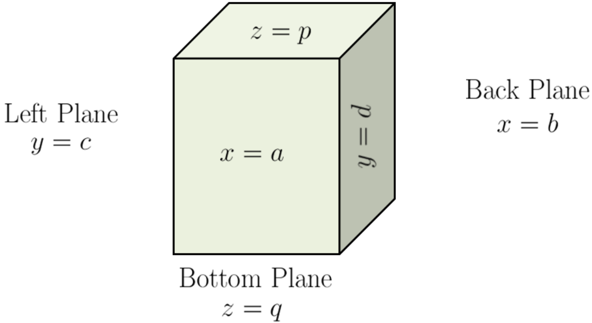
$$\iiint_E f\, dV = \int_p^q \int_{g_9(z)}^{g_{10}(z)} \int_{h_9(y,z)}^{h_{10}(y,z)} f(x,y,z)\, dx\, dy\, dz$$

OR

$$\iiint_E f\, dV = \int_a^b \int_{g_{11}(x)}^{g_{12}(x)} \int_{h_{11}(x,z)}^{h_{12}(x,z)} f(x,y,z)\, dy\, dz\, dx$$

To compute an iterated integral, compute the ”inner integral” first, then compute the ”middle integral”, then compute the ”outer integral.”

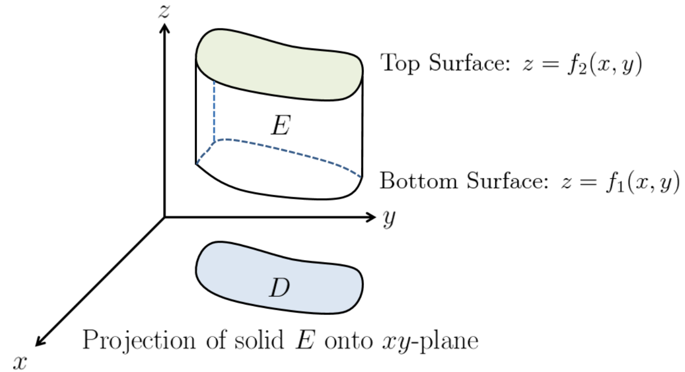
Volume of a Box using Triple Integrals



$$\iiint_E dV = \int_a^b \int_c^d \int_p^q dz\,dy\,dx = \int_c^d \int_a^b \int_p^q dz\,dx\,dy = \int_a^b \int_p^q \int_c^d dy\,dz\,dx$$

$$= \int_p^q \int_a^b \int_c^d dy\,dx\,dz = \int_c^d \int_p^q \int_a^b dx\,dz\,dy = \int_p^q \int_c^d \int_a^b dx\,dy\,dz$$

Volume of a z-Simple Solid using Triple Integrals

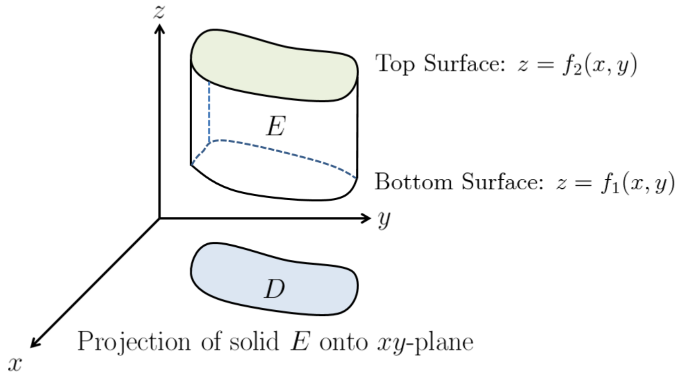


$$\iiint_E dV = \iint_D \left[\int_{\text{Bottom BS of } E}^{\text{Top BS of } E} dz \right] dA = \iint_D \left[\int_{f_1(x,y)}^{f_2(x,y)} dz \right] dA$$

BS $\equiv$ **B**oundary **S**urface

REMARK: After computing the inner integral, if the double integral is too hard to compute, rewrite double integral in **polar coordinates**.

Volume of a z-Simple Solid using Triple Integrals

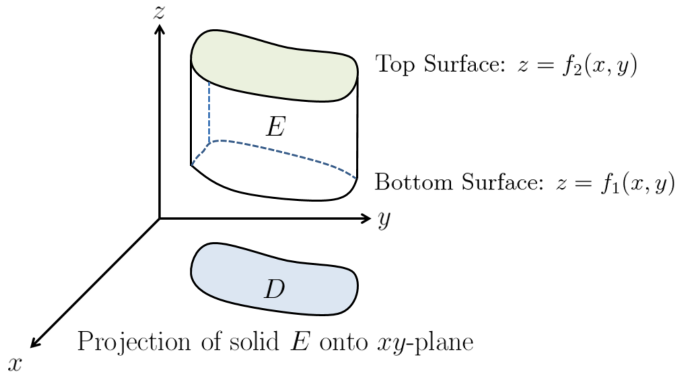


$$\iiint_E dV = \int_{\text{Smallest } x\text{-coord in } D}^{\text{Largest } x\text{-coord in } D} \int_{\text{Btm BC of } D}^{\text{Top BC in } D} \int_{\text{Bottom BS of } E}^{\text{Top BS of } E} dz\, dy\, dx$$

<u>REMARK:</u> This iterated triple integral is for illustration only!

It's far less error-prone to start with $\displaystyle\iint_D \left[\int_{\text{Bottom BS of } E}^{\text{Top BS of } E} dz \right] dA$

Volume of a z-Simple Solid using Triple Integrals

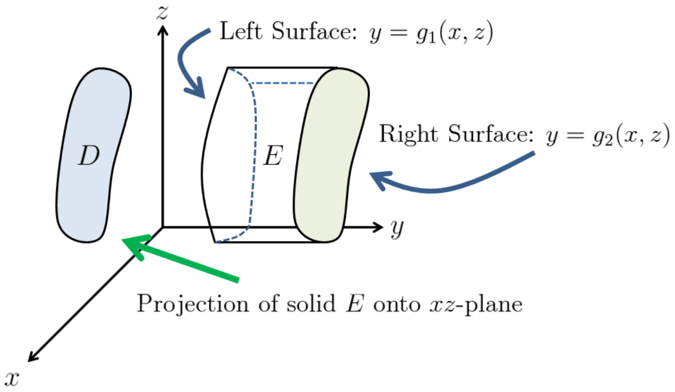


$$\iiint_E dV = \int_{\text{Smallest } y\text{-coord in } D}^{\text{Largest } y\text{-coord in } D} \int_{\text{Left BC of } D}^{\text{Right BC in } D} \int_{\text{Bottom BS of } E}^{\text{Top BS of } E} dz\, dx\, dy$$

<u>REMARK:</u> This iterated triple integral is for illustration only!

It's far less error-prone to start with $\iint_D \left[\int_{\text{Bottom BS of } E}^{\text{Top BS of } E} dz \right] dA$

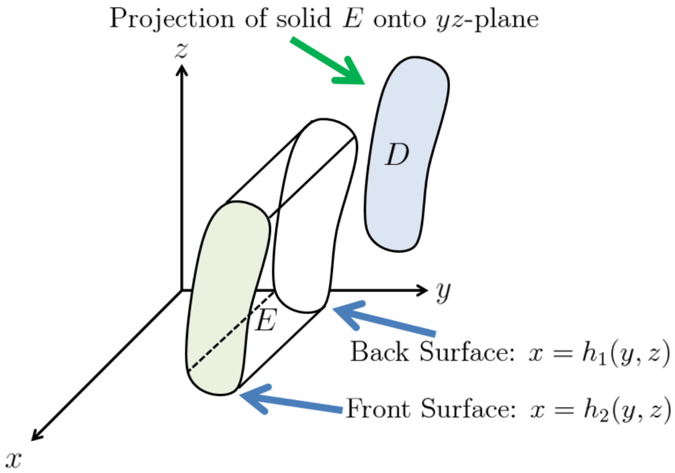
Volume of a y-Simple Solid (Optional)



$$\iiint_E dV = \iint_D \left[\int_{\text{Left BS of } E}^{\text{Right BS of } E} dy \right] dA = \iint_D \left[\int_{g_1(x,z)}^{g_2(x,z)} dy \right] dA$$

BS $\equiv$ **B**oundary **S**urface

Volume of a x-Simple Solid (Optional)



$$\iiint_E dV = \iint_D \left[\int_{\text{Back BS of } E}^{\text{Front BS of } E} dx \right] dA = \iint_D \left[\int_{h_1(y,z)}^{h_2(y,z)} dx \right] dA$$

BS $\equiv$ **B**oundary **S**urface

Fin.