Triple Integrals: Cylindrical & Spherical Coordinates

Calculus III

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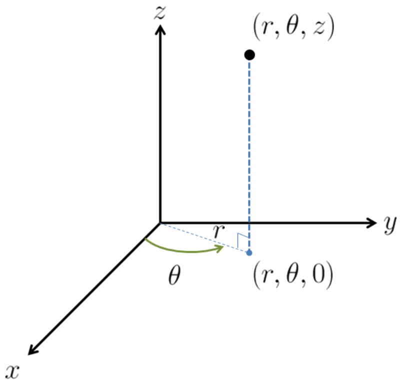
TTU

27 October 2014

PART I:

SETUP OF TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Cylindrical Coordinates



COORDINATE	INTERPRETATION	REMARK(S)
r	$\lvert r\rvert \equiv$ Distance from origin to point	$r \in \mathbb{R}$ r can be negative
θ	Angle swept CCW from positive x-axis	$\theta \in \mathbb{R}$ Often $\theta \in [0, 2\pi]$
z	$\lvert z\rvert \equiv$ Distance from xy-plane to point	$z \in \mathbb{R}$

Cylindrical Coordinates

Proposition

(Rectangular → Cylindrical)

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

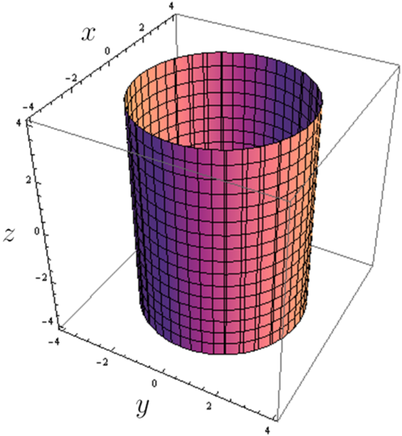
Proposition

(Cylindrical → Rectangular)

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$
$$z = z$$

REMARK: The focus will be converting Rectangular → Cylindrical (top box).

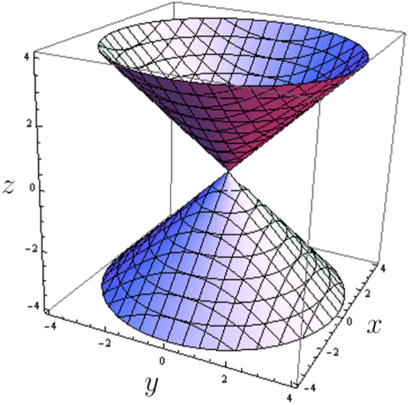
Surfaces with Simple Cylindrical Forms



(Circular) Cylinder

Rectangular Form: $x^2 + y^2 = k^2$
Cylindrical Form: $r = k$
$(k \in \mathbb{R})$

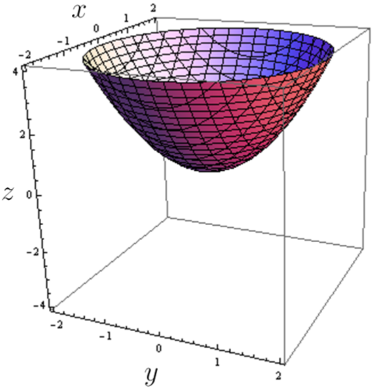
Surfaces with Simple Cylindrical Forms



(Circular) Cone

Rectangular Form: $x^2 + y^2 = z^2$
Cylindrical Form: $r = z$

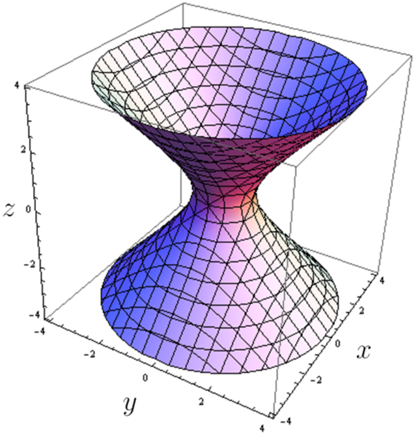
Surfaces with Simple Cylindrical Forms



(Circular) Paraboloid

Rectangular Form: $x^2 + y^2 = kz$
Cylindrical Form: $r^2 = kz$
$(k \in \mathbb{R})$

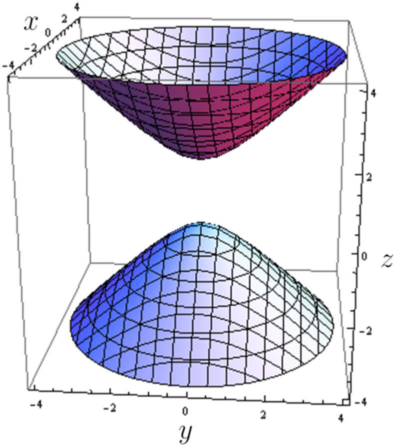
Surfaces with Simple Cylindrical Forms



(Circular) Hyperboloid of 1 Sheet

Rectangular Form: $x^2 + y^2 - z^2 = 1$
Cylindrical Form: $r^2 = z^2 + 1$

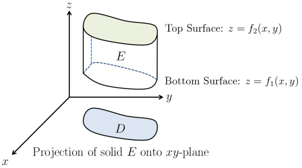
Surfaces with Simple Cylindrical Forms



(Circular) Hyperboloid of 2 Sheets

Rectangular Form: $x^2 + y^2 - z^2 = -1$
Cylindrical Form: $r^2 = z^2 - 1$

Triple Integrals in Cylindrical Coordinates



Proposition

(Triple Integral in Cylindrical Coordinates)

Let $f(x,y,z) \in C(E)$, where solid $E \subset \mathbb{R}^3$ is z-simple s.t. its proj. D is r-simple.

$$\iiint_E f\, dV \overset{CYL}{=} \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \int_{\text{Inner BC of } D}^{\text{Outer BC of } D} \int_{\text{Btm BS in cyl. form}}^{\text{Top BS in cyl. form}} f\ r\ dz\ dr\ d\theta$$

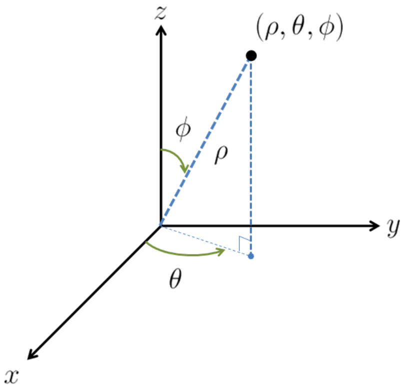
$$= \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r\cos\theta, r\sin\theta)}^{f_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta$$

If region D only has an outer BC, then the inner BC is the **pole** $(r = 0)$.
REMARK: **Always** integrate in the order $dz\ dr\ d\theta$.

PART II:

SETUP OF TRIPLE INTEGRALS IN SPHERICAL COORDINATES

Spherical Coordinates



COORDINATE	INTERPRETATION	REMARK(S)
ρ	Distance from origin to point	$\rho \geq 0$
θ	Angle swept CCW from positive x-axis	$\theta \in \mathbb{R}$ Often $\theta \in [0, 2\pi]$
ϕ	Angle swept from positive z-axis	$\phi \in [0, \pi]$

$\rho \equiv$ "rho" $\theta \equiv$ "theta" $\phi \equiv$ "phi"

"The Sphere's Notational Curse"

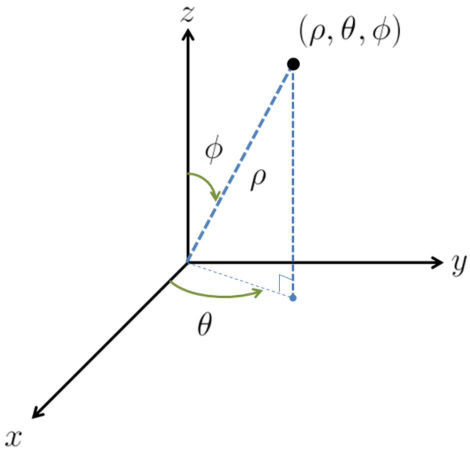
Unfortunately, other books use different notations for spherical coordinates:

MATH	PHYSICS	GIS/CARTOGRAPHY	ASTRONOMY
(ρ, θ, ϕ)	(ρ, ϕ, θ)	(It's Complicated)	(It's Complicated)
(ρ, ϑ, ϕ)	(ρ, ϕ, ϑ)		
(ρ, θ, φ)	(ρ, φ, θ)		
$(\rho, \vartheta, \varphi)$	$(\rho, \varphi, \vartheta)$		
(r, θ, ϕ)	(r, ϕ, θ)		
(r, ϑ, ϕ)	(r, ϕ, ϑ)		
(r, θ, φ)	(r, φ, θ)		
(r, ϑ, φ)	(r, φ, ϑ)		

Notations in red are frowned upon by your instructor.....

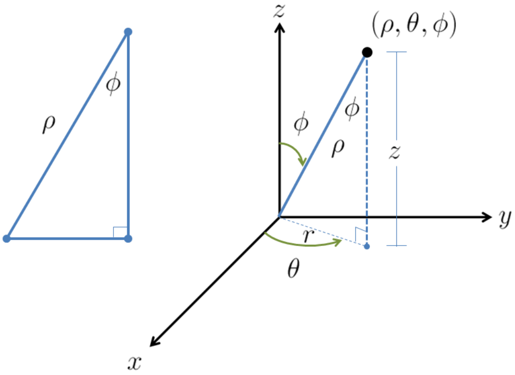
The convention used going forward is (ρ, θ, ϕ).

Converting Rectangular → Spherical (Derivation)



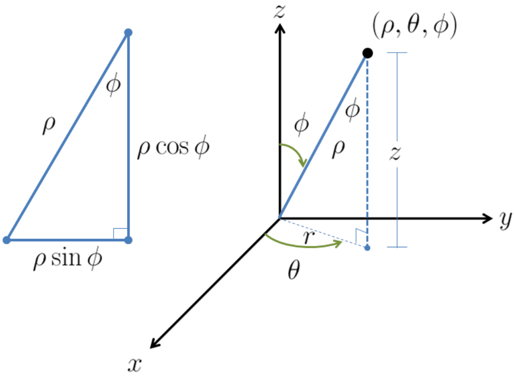
What are x, y, z in terms of ρ, ϕ, θ???

Converting Rectangular → Spherical (Derivation)

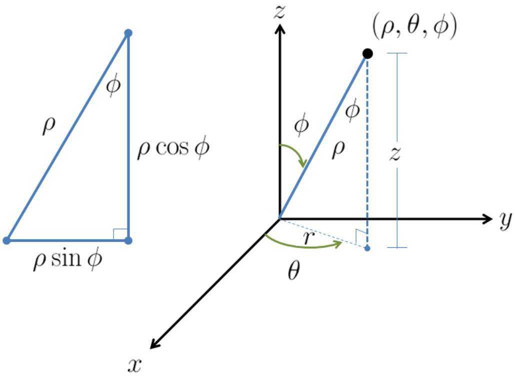


The legs of the right triangle shown are, by definition, r and z.
However, the legs can also be written in terms of hypotenuse ρ and angle ϕ.

Converting Rectangular → Spherical (Derivation)

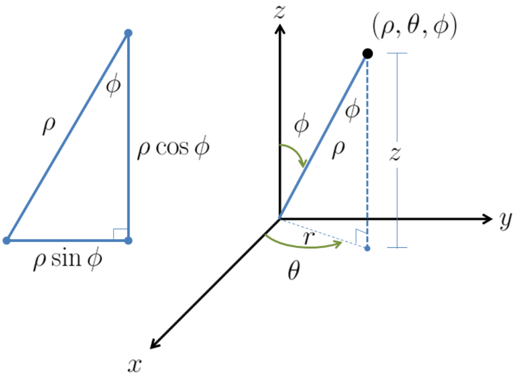


Converting Rectangular → Spherical (Derivation)



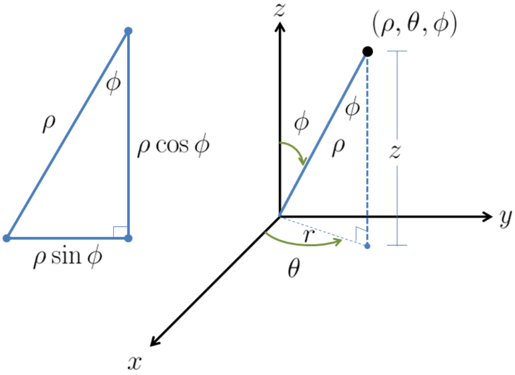
$z = \rho\cos\phi$
$r = \rho\sin\phi$

Converting Rectangular → Spherical (Derivation)



$$\begin{aligned} z &= \rho \cos \phi \\ r &= \rho \sin \phi \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= r \cos \theta = (\rho \sin \phi) \cos \theta \\ y &= r \sin \theta = (\rho \sin \phi) \sin \theta \end{aligned}$$

Converting Rectangular → Spherical (Derivation)



$$z = \rho\cos\phi \qquad \qquad x = r\cos\theta = (\rho\sin\phi)\cos\theta \qquad \qquad x = \rho\sin\phi\cos\theta$$
$$r = \rho\sin\phi \quad \Longrightarrow \quad y = r\sin\theta = (\rho\sin\phi)\sin\theta \quad \Longrightarrow \quad y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Converting Rectangular ↔ Spherical

Proposition

(Rectangular → Spherical)

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

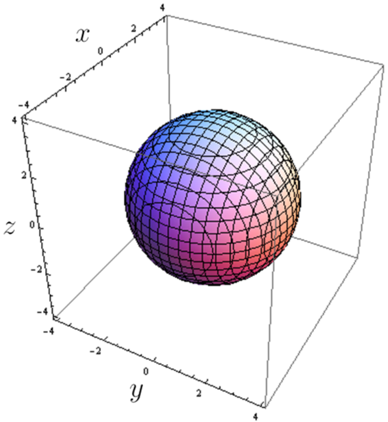
Proposition

(Spherical → Rectangular)

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

<u>REMARK:</u> The focus will be converting Rectangular → Spherical (top box).
<u>REMARK:</u> The angle ϕ is not to be confused with the empty set $\emptyset$.

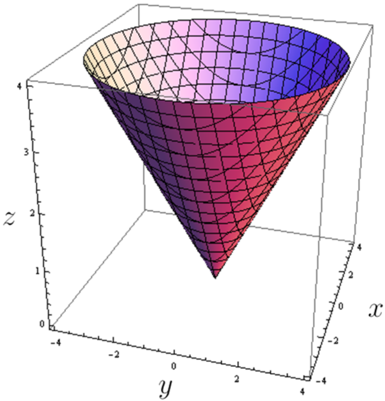
Surfaces with Simple Spherical Forms



Sphere

Rectangular Form: $x^2 + y^2 + z^2 = k^2$
Spherical Form: $\rho = k$
$(k > 0)$

Surfaces with Simple Spherical Forms

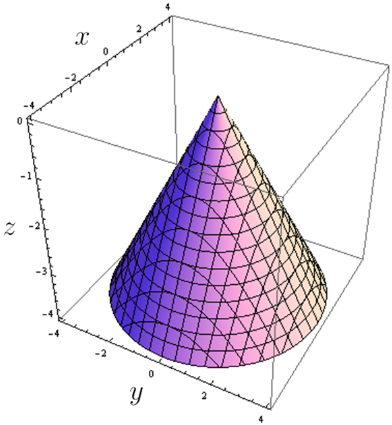


Half-Cone

Spherical Form: $\phi = k$

$$\left(0 < k < \frac{\pi}{2}\right)$$

Surfaces with Simple Spherical Forms

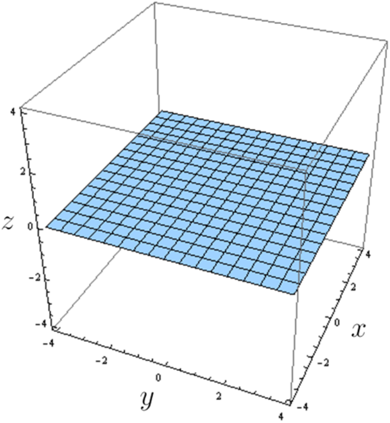


Half-Cone

Spherical Form: $\phi = k$

$$\left(\frac{\pi}{2} < k < \pi\right)$$

Surfaces with Simple Spherical Forms



xy-plane

Rectangular Form: $z = 0$

Spherical Form: $\phi = \frac{\pi}{2}$

Spherical Forms of Planes

WORKED EXAMPLE: Write the plane $z = 4$ in spherical form.

$$z = 4 \implies \rho\cos\phi = 4 \implies \rho = \frac{4}{\cos\phi} \implies \boxed{\rho = 4\sec\phi}$$

WORKED EXAMPLE: Write the plane $y = 4$ in spherical form.

$$y = 4 \implies \rho\sin\phi\sin\theta = 4 \implies \rho = \frac{4}{\sin\phi\sin\theta} \implies \boxed{\rho = 4\csc\phi\csc\theta}$$

WORKED EXAMPLE: Write the plane $x = 4$ in spherical form.

$$x = 4 \implies \rho\sin\phi\cos\theta = 4 \implies \rho = \frac{4}{\sin\phi\cos\theta} \implies \boxed{\rho = 4\csc\phi\sec\theta}$$

WORKED EXAMPLE: Write the plane $x + 2y + 3z = 4$ in spherical form.

$$x + 2y + 3z = 4 \implies (\rho\sin\phi\cos\theta) + 2(\rho\sin\phi\sin\theta) + 3(\rho\cos\phi) = 4$$

$$\implies \boxed{\rho = \frac{4}{\sin\phi\cos\theta + 2\sin\phi\sin\theta + 3\cos\phi}}$$

Spherical Forms of Cones

WORKED EXAMPLE: Write the half-cone $z = \sqrt{x^2 + y^2}$ in spherical form.

$z = \sqrt{x^2 + y^2} \implies z \geq 0 \implies \phi \in [0, \pi/2]$

Apply conversion $\{x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi\}$ to the half-cone and simplify:

$$\begin{aligned}
z &= \sqrt{x^2 + y^2} \\
\rho\cos\phi &= \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} \\
&= \sqrt{\rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)} \\
&= \sqrt{\rho^2\sin^2\phi} && \text{(Trig Identity)} \\
&= |\rho\sin\phi| && \left(\sqrt{x^2} := |x|\right) \\
&= \rho\sin\phi && (\text{Since } \rho \geq 0 \text{ and } \phi \geq 0)
\end{aligned}$$

$\therefore \rho\cos\phi = \rho\sin\phi$

$$\begin{aligned}
&\implies \cos\phi = \sin\phi && (\text{Since } \rho = 0 \text{ describes the origin, not a cone}) \\
&\implies \tan\phi = 1 && (\sin\phi = 0 \implies \phi \in \{0, \pi\} \text{ describes } z\text{-axis, not a cone}) \\
&\implies \boxed{\phi = \pi/4} && (\text{Since } 0 \leq \phi \leq \pi/2)
\end{aligned}$$

Triple Integrals in Spherical Coordinates

Proposition

(Triple Integral in Spherical Coordinates)

Let $f(x,y,z) \in C(E)$ s.t. $E \subset \mathbb{R}^3$ is a ***closed*** *&* ***bounded*** *solid . Then:*

$$\iiint_E f\,dV \overset{SPH}{=} \int_{\text{Smallest } \theta\text{-val in } E}^{\text{Largest } \theta\text{-val in } E} \int_{\text{Smallest } \phi\text{-val in } E}^{\text{Largest } \phi\text{-val in } E} \int_{\text{Inside BS of } E}^{\text{Outside BS of } E} f\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

— OR EQUVIALENTLY —

$$\iiint_E f\,dV \overset{SPH}{=} \int_{\text{Smallest } \phi\text{-val in } E}^{\text{Largest } \phi\text{-val in } E} \int_{\text{Smallest } \theta\text{-val in } E}^{\text{Largest } \theta\text{-val in } E} \int_{\text{Inside BS of } E}^{\text{Outside BS of } E} f\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\ \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

REMARK: If there's only one surface, treat **origin** $(\rho = 0)$ as **inner BS** of E.

REMARK: Setting up is harder since projection of E on xy-plane is **useless**!

Fin

Fin.